

ON LINEAR RELATIONS AMONG TOTALLY ODD MULTIPLE ZETA VALUES RELATED TO PERIOD POLYNOMIALS

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ABSTRACT. An interesting connection between the theory of elliptic modular forms and multiple zeta values (MZVs) was first discovered by Don Zagier in the case of depth 2. We will provide this connection for arbitrary depths through the study of linear relations among MZVs at the sequences indexed by odd integers greater than 1, modulo lower depth and $\zeta(2)$. This work is motivated by a certain dimension conjecture proposed by Francis Brown. We finally present an affirmative answer to his dimension conjecture in the case of depth 4.

1. INTRODUCTION

In this paper, we will be interested in the relationship between \mathbb{Q} -linear relations among multiple zeta values

$$\zeta(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} \quad (n_1, \dots, n_{r-1} \in \mathbb{Z}_{>0}, n_r \in \mathbb{Z}_{>1})$$

(MZVs for short) and period polynomials of the elliptic modular form for the full modular group $\Gamma_1 := \mathrm{PSL}_2(\mathbb{Z})$. This interesting connection was first discovered in the case of depth 2 by Zagier [13, §8] (see also [14, §3]), and then investigated in depth by Gangl, Kaneko and Zagier [5]. We will present this connection for arbitrary depths through the study of totally odd MZVs which are MZVs at the sequences indexed by odd integers greater than 1, modulo all MZVs of lower depth and the ideal generated by $\zeta(2)$.

We wish to study the relations among totally odd MZVs to reduce the dimension of the \mathbb{Q} -vector space spanned by totally odd MZVs to the theoretically predicted one (see Conjecture 2.1). An important framework of this study has been done by Brown [3, Section 10]. This uses the notion of his celebrated paper [2]: Brown's

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operator D_m , which is an infinitesimal version of the motivic coaction on the motivic MZVs

$$\zeta^{\mathfrak{m}}(n_1, \dots, n_r),$$

preserves a certain filtration structure, and it gives rise to a certain derivation ∂_m . With this Brown defined key matrices $C_{N,r}$ (Definition 2.3) whose right annihilator gives a linear relation among totally odd MZVs of weight N and depth r (see §2.3). More strongly, it can be shown that all relations among totally odd motivic MZVs of weight N and depth r (modulo lower depth) are obtained from right annihilators of the matrix $C_{N,r}$ (as a consequence of Proposition 2.2). Thus, together with the period map sending $\zeta^{\mathfrak{m}}(n_1, \dots, n_r)$ to $\zeta(n_1, \dots, n_r)$, we have:

Proposition 1.1. *The number of linear relations among totally odd MZVs of weight N and depth r (modulo lower depth and $\zeta(2)$) is bounded by the dimension of the space of right annihilators of $C_{N,r}$.*

Thanks to this proposition, the problem of finding an upper bound of the dimension of the \mathbb{Q} -vector space $\mathcal{Z}_{N,r}^{\text{odd}}$ spanned by all totally odd MZVs of weight N and depth r can be reduced to the solution of an elementary problem of linear algebra for the matrix $C_{N,r}$. Indeed, it turns out that $\dim_{\mathbb{Q}} \mathcal{Z}_{N,r}^{\text{odd}}$ is less than or equal to $\text{rank } C_{N,r}$ (Corollary 2.5). As the coefficients of $C_{N,r}$ are complicated, it seems to be hard to give an exact value of $\text{rank } C_{N,r}$ in general. Some information about it however can be obtained by looking at left annihilators of $C_{N,r}$: for example, all left annihilators of $C_{N,2}$ can be characterised by restricted even period polynomials of degree $N-2$, which was first shown by Baumard and Schneps [4]. Here the restricted even period polynomial of degree $N-2$ is defined as an even, homogeneous polynomial $p(x_1, x_2) \in \mathbb{Q}[x_1, x_2]_{(N-2)}$ of degree $N-2$ such that $p(x_1, 0) = 0$ and

$$(1.1) \quad p(x_1, x_2) - p(x_2 - x_1, x_2) + p(x_2 - x_1, x_1) = 0.$$

The important fact about this polynomial is that the dimension of its associated vector space over \mathbb{Q} coincides with that of the \mathbb{C} -vector space of cusp forms for Γ_1 , which follows from, known as Eichler-Shimura-Manin correspondence, the results of [11, §1.1] and [5, §5]. Therefore, since the matrix $C_{N,r}$ is square, the above characterisation provides a lower bound of the number of relations, and also an upper bound of $\text{rank } C_{N,2}$. It is vital to note that relating with period polynomials is actually our fundamental principle of computing an upper bound of $\text{rank } C_{N,r}$.

Our first goal (§3) is to relate the relations among totally odd MZVs and the restricted even period polynomial through the study of a certain matrix $E_{N,r}$ (Definition 3.2). We start from showing that the matrix $E_{N,r}$ is a simple factor of the matrix $C_{N,r}$ (Proposition 3.3), which gives the following proposition:

Proposition 1.2. *The space of right annihilators of $E_{N,r}$ is contained in that of $C_{N,r}$.*

Combining this with Proposition 2.4, one can find that the right annihilator of $E_{N,r}$ gives a linear relations among totally odd MZVs of weight N and depth r . As in the case of $C_{N,r}$, it seems to be difficult to determine the dimension of the space of right annihilators of $E_{N,r}$. However, by looking at left annihilators of the matrix $E_{N,r}$, one can meet a mysterious correspondence to the restricted even period polynomial, and hence give a lower bound of the dimension of the space of left annihilators of $E_{N,r}$. More precisely, let $\mathbf{W}_{N,r}$ be the space of even, homogeneous polynomials $p(x_1, \dots, x_r) \in \mathbb{Q}[x_1, \dots, x_r]_{(N-r)}$ of degree $N - r$ such that $p(x_1, \dots, x_r)|_{x_i=0} = 0$ for $1 \leq i \leq r$ and

$$p(x_1, \dots, x_r) - p(x_2 - x_1, x_2, \underbrace{x_3, \dots, x_r}_{r-2}) + p(x_2 - x_1, x_1, \underbrace{x_3, \dots, x_r}_{r-2}) = 0.$$

Each element in $\mathbf{W}_{N,2}$ is just a restricted even period polynomial, and the defining relation in the space $\mathbf{W}_{N,r}$ only uses the relation (1.1), so that the exact dimension of the space of $\mathbf{W}_{N,r}$ can be easily obtained. Our first main result is as follows (Proposition 3.4 and Theorem 3.6).

Theorem 1.3. (1) *There is a one-to-one correspondence between $\mathbf{W}_{N,2}$ and the space of left annihilators of the matrix $E_{N,2}(= C_{N,2})$.*

(2) *For $r \geq 3$, there is an injective map from $\mathbf{W}_{N,r}$ to the space of left annihilators of the matrix $E_{N,r}$.*

Theorem 1.3 (1) is the result of Baumard and Schneps, which will be reproved in §3.3 for completeness. The map in the statement of Theorem 1.3 (2) is quite simple and guessed from numerical experiments (so we have no idea to explain this map theoretically). It is interesting to note that this map is conjecturally surjective, i.e., all left annihilators of the matrix $E_{N,r}$ can be characterised by restricted even period polynomials. We were not able to prove this conjecture. However, Theorem 1.3 helps us give a new upper bound of the dimension of the space $\mathcal{Z}_{N,r}^{\text{odd}}$.

Proposition 1.2 and Theorem 1.3 enable us to give a lower bound of the number of linearly independent relations among totally odd MZVs of weight N and depth r obtained from right annihilators of $E_{N,r}$, since $E_{N,r}$ is a square matrix. We notice that these relations do not suffice to obtain the predicted number of relations whenever $r \geq 3$. To obtain more relations, we use an algebraic structure of the space $\bigoplus_{N,r \geq 0} \mathcal{Z}_{N,r}^{\text{odd}}$. This space forms a commutative \mathbb{Q} -algebra with respect to Hoffman's harmonic product (modulo lower depth). Then one can obtain linear relations among totally odd MZVs of weight N and depth r from the right annihilator of the matrix $E_{N',r'}$ for $(N', r') < (N, r)$ by multiplying totally odd MZVs of weight $N - N'$ and depth $r - r'$. In addition, it turns out that the coefficient vector of such relations becomes a right annihilator of the matrix $C_{N,r}$ when $r = 4$ (Corollary 4.2). This, together with a small lemma about the shuffle algebra, shows our second main result.

Theorem 1.4. *We have*

$$\sum_{N>0} \text{rank } C_{N,4} x^N \leq \mathbb{O}(x)^4 - 3\mathbb{O}(x)^2 \mathbb{S}(x) + \mathbb{S}(x)^2,$$

where $\mathbb{O}(x) = \frac{x^3}{1-x^2}$, $\mathbb{S}(x) = \frac{x^{12}}{(1-x^4)(1-x^6)}$ and $\sum_{N>0} a_N x^N \leq \sum_{N>0} b_N x^N$ means $a_N \leq b_N$ for all $N > 0$.

We note that the predicted upper bound for $\dim_{\mathbb{Q}} \mathcal{Z}_{N,r}^{\text{odd}}$ can be obtained by using the results of Goncharov [6] when $r = 2, 3$. As a consequence of Theorem 1.4, we obtain a new upper bound of $\dim_{\mathbb{Q}} \mathcal{Z}_{N,4}^{\text{odd}}$, which is an affirmative answer to Brown's conjecture (Conjecture 2.1) in the case of depth 4.

Corollary 1.5. *We have*

$$\sum_{N>0} \dim_{\mathbb{Q}} \mathcal{Z}_{N,4}^{\text{odd}} x^N \leq \mathbb{O}(x)^4 - 3\mathbb{O}(x)^2 \mathbb{S}(x) + \mathbb{S}(x)^2.$$

The contents are as follows. In Section 2, we clarify the totally odd MZVs conjecture and notations, and give a brief review of Brown's works in [3], including Proposition 1.1. Section 3 studies the matrix $E_{N,r}$. We shall give proofs of Proposition 1.2 and Theorem 1.3. Section 4 is devoted to proving Theorem 1.4. This proof need certain lemmas, which will be proven in the Appendix.

2. PRELIMINARIES

2.1. Totally odd multiple zeta values conjecture. In this subsection, we give a precise definition of the totally odd MZVs and state its dimension conjecture proposed by Brown [3].

The MZV is defined for $\mathbf{n} = (n_1, \dots, n_r) \in \mathbb{Z}_{>0}^r$ with $n_r \geq 2$ by

$$\zeta(\mathbf{n}) = \zeta(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}}.$$

As usual, we call $n_1 + \dots + n_r$ ($=: \text{wt}(\mathbf{n})$) the weight and r ($=: \text{dep}(\mathbf{n})$) the depth, and $1 \in \mathbb{Q}$ is regarded as the unique MZV of weight 0 and depth 0. Let \mathcal{Z} be the MZV algebra $\bigoplus_{N \geq 0} \mathcal{Z}_N$, where \mathcal{Z}_N is the \mathbb{Q} -vector space spanned by all MZVs of weight N , and \mathfrak{D} the depth filtration on \mathcal{Z} :

$$\mathfrak{D}_0 \mathcal{Z} = \mathbb{Q} \subset \mathfrak{D}_1 \mathcal{Z} \subset \dots \subset \mathfrak{D}_r \mathcal{Z} := \langle \zeta(\mathbf{n}) \mid \text{dep}(\mathbf{n}) \leq r \rangle_{\mathbb{Q}} \subset \dots$$

The MZV algebra becomes a filtered algebra with \mathfrak{D} . Let $\mathcal{Z}_{N,r}$ be the \mathbb{Q} -vector space of the weight N and depth r part of the bigraded \mathbb{Q} -algebra $\text{gr}^{\mathfrak{D}}(\mathcal{Z}/\zeta(2)\mathcal{Z}) = \bigoplus_{N,r \geq 0} \mathcal{Z}_{N,r}$ and $\zeta_{\mathfrak{D}}(\mathbf{n})$ denote the equivalence class of $\zeta(\mathbf{n})$ of weight N and depth r in $\mathcal{Z}_{N,r}$, called the depth-graded MZV. Then the \mathbb{Q} -vector space $\mathcal{Z}_{N,r} = \mathfrak{D}_r \mathcal{Z}_N / (\mathfrak{D}_{r-1} \mathcal{Z}_N + \mathfrak{D}_r \mathcal{Z}_N \cap \zeta(2)\mathcal{Z})$ is spanned by all depth-graded MZVs of weight N and depth r . Let us call $\zeta_{\mathfrak{D}}(n_1, \dots, n_r)$ the totally odd MZV when all n_i are odd (≥ 3). The \mathbb{Q} -vector subspace of $\mathcal{Z}_{N,r}$ spanned by all totally odd MZVs of weight N and depth r is denoted by

$$\mathcal{Z}_{N,r}^{\text{odd}} = \langle \zeta_{\mathfrak{D}}(\mathbf{n}) \in \mathcal{Z}_{N,r} \mid \mathbf{n} \in S_{N,r} \rangle_{\mathbb{Q}},$$

where $S_{N,r}$ is the set of totally odd indices of weight N and depth r :

$$S_{N,r} = \{(n_1, \dots, n_r) \in \mathbb{Z}^r \mid n_1 + \dots + n_r = N, n_1, \dots, n_r \geq 3 : \text{odd}\}.$$

We set $\mathcal{Z}_{0,0}^{\text{odd}} = \mathbb{Q}$. Notice that the number of elements of the set $S_{N,r}$ obviously gives a trivial upper bound $\dim_{\mathbb{Q}} \mathcal{Z}_{N,r}^{\text{odd}} \leq |S_{N,r}|$, and hence $\mathcal{Z}_{N,r}^{\text{odd}} = \{0\}$ whenever $N \not\equiv r \pmod{2}$. We now state the totally odd MZVs conjecture.

Conjecture 2.1. ([3, Eq. (10.4)]) *The generating function of the dimension of the space $\mathcal{Z}_{N,r}^{\text{odd}}$ is given by*

$$(2.1) \quad \sum_{N,r \geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_{N,r}^{\text{odd}} x^N y^r \stackrel{?}{=} \frac{1}{1 - \mathbb{O}(x)y + \mathbb{S}(x)y^2},$$

where $\mathbb{O}(x) = \frac{x^3}{1-x^2} = x^3 + x^5 + x^7 + \dots$ and $\mathbb{S}(x) = \frac{x^{12}}{(1-x^4)(1-x^6)} = x^{12} + x^{16} + x^{18} + \dots$.

Let us give a few examples. By expanding the right-hand side of (2.1) at $y = 0$, one can derive

$$\begin{aligned} \sum_{N>0} \dim_{\mathbb{Q}} \mathcal{Z}_{N,2}^{\text{odd}} x^N &\stackrel{?}{=} \mathbb{O}(x)^2 - \mathbb{S}(x), \quad \sum_{N>0} \dim_{\mathbb{Q}} \mathcal{Z}_{N,3}^{\text{odd}} x^N \stackrel{?}{=} \mathbb{O}(x)^3 - 2\mathbb{O}(x)\mathbb{S}(x), \\ \sum_{N>0} \dim_{\mathbb{Q}} \mathcal{Z}_{N,4}^{\text{odd}} x^N &\stackrel{?}{=} \mathbb{O}(x)^4 - 3\mathbb{O}(x)^2\mathbb{S}(x) + \mathbb{S}(x)^2 \end{aligned}$$

and so on. Since $\mathbb{O}(x)^r = \sum_{N>0} |\mathbb{S}_{N,r}| x^N$ and the coefficient of x^N in $\mathbb{S}(x)$ coincides with the dimension of the space of cusp forms of weight N for Γ_1 , Conjecture 2.1 suggests that all linear relations among totally odd MZVs relate to cusp forms. In the cases of $r = 2$ and $r = 3$, from the results of Goncharov [6, Theorems 2.4 and 2.5] (see also [9, Proposition 18], [10, Theorem 19]) on the number of algebra generators of the depth-graded \mathbb{Q} -algebra $\text{gr}^{\mathfrak{D}}(\mathcal{Z}/\zeta(2)\mathcal{Z})$, we find

$$\sum_{N>0} \dim_{\mathbb{Q}} \mathcal{Z}_{N,2}^{\text{odd}} x^N \leq \mathbb{O}(x)^2 - \mathbb{S}(x) \quad \text{and} \quad \sum_{N>0} \dim_{\mathbb{Q}} \mathcal{Z}_{N,3}^{\text{odd}} x^N \leq \mathbb{O}(x)^3 - 2\mathbb{O}(x)\mathbb{S}(x).$$

Remark. To obtain the above inequalities, we have used $\dim_{\mathbb{Q}} \mathcal{Z}_{N,r}^{\text{odd}} \leq \dim_{\mathbb{Q}} \mathcal{Z}_{N,r}$. When $r = 2$, from the results of Gangl, Kaneko and Zagier [5, Theorem 2], one can deduce $\mathcal{Z}_{N,2}^{\text{odd}} = \mathcal{Z}_{N,2}$. However we do not know $\mathcal{Z}_{N,3}^{\text{odd}} \stackrel{?}{=} \mathcal{Z}_{N,3}$.

2.2. Notations. To clarify the meaning of the left (or right) annihilator of our matrices described in the Introduction, we fix notations. For integers N, r with $|\mathbb{S}_{N,r}| > 0$, the notation

$$M = \left(m_{\substack{(m_1, \dots, m_r) \\ (n_1, \dots, n_r)}} \right)_{\substack{(m_1, \dots, m_r) \in \mathbb{S}_{N,r} \\ (n_1, \dots, n_r) \in \mathbb{S}_{N,r}}}$$

means that M is a $|\mathbb{S}_{N,r}| \times |\mathbb{S}_{N,r}|$ matrix with $m_{\substack{(m_1, \dots, m_r) \\ (n_1, \dots, n_r)}} \in \mathbb{Z}$ in the $((m_1, \dots, m_r), (n_1, \dots, n_r))$ entry (i.e. rows and columns are indexed by (m_1, \dots, m_r) and (n_1, \dots, n_r) in the set $\mathbb{S}_{N,r}$ respectively). For convenience we regard M as an empty matrix when $|\mathbb{S}_{N,r}| = 0$ (i.e. $\text{rank } M = 0$). Let us denote by $\mathbf{Vect}_{N,r}$ the $|\mathbb{S}_{N,r}|$ -dimensional vector space over \mathbb{Q} of row vectors $(a_{n_1, \dots, n_r})_{(n_1, \dots, n_r) \in \mathbb{S}_{N,r}}$ indexed by totally odd indices $(n_1, \dots, n_r) \in \mathbb{S}_{N,r}$ with rational coefficients:

$$\mathbf{Vect}_{N,r} = \{ (a_{n_1, \dots, n_r})_{(n_1, \dots, n_r) \in \mathbb{S}_{N,r}} \mid a_{n_1, \dots, n_r} \in \mathbb{Q} \}.$$

We identify the matrix M with its induced linear map on $\mathbf{Vect}_{N,r}$

$$\begin{aligned} M : \mathbf{Vect}_{N,r} &\longrightarrow \mathbf{Vect}_{N,r} \\ v &\longmapsto v \cdot M, \end{aligned}$$

so that, for $v = (a_{n_1, \dots, n_r})_{(n_1, \dots, n_r) \in S_{N,r}} \in \mathbf{Vect}_{N,r}$

$$M(v) = \left(\sum_{(m_1, \dots, m_r) \in S_{N,r}} a_{m_1, \dots, m_r} m \binom{m_1, \dots, m_r}{n_1, \dots, n_r} \right)_{(n_1, \dots, n_r) \in S_{N,r}}.$$

It is clear that a row vector $v = (a_{n_1, \dots, n_r})_{(n_1, \dots, n_r) \in S_{N,r}} \in \mathbf{Vect}_{N,r}$ satisfies $M(v) = 0$ if and only if $\sum_{(m_1, \dots, m_r) \in S_{N,r}} a_{m_1, \dots, m_r} m \binom{m_1, \dots, m_r}{n_1, \dots, n_r} = 0$ for all $(n_1, \dots, n_r) \in S_{N,r}$. Hereafter, we use the notion $v \in \ker M \subset \mathbf{Vect}_{N,r}$ (resp. $\ker {}^t M \subset \mathbf{Vect}_{N,r}$) instead of saying v (resp. ${}^t v$) is a left (resp. right) annihilator of the matrix M , where we understand ${}^t M = \left(m \binom{n_1, \dots, n_r}{m_1, \dots, m_r} \right)_{\substack{(m_1, \dots, m_r) \in S_{N,r} \\ (n_1, \dots, n_r) \in S_{N,r}}}$.

2.3. Linear relations among totally odd MZVs. The motivic multiple zeta value $\zeta^{\mathfrak{m}}(n_1, \dots, n_r)$ plays a key role in detecting Conjecture 2.1. This is an element of a certain \mathbb{Q} -algebra $\mathcal{H} = \bigoplus_{N \geq 0} \mathcal{H}_N$ constructed by Brown [2, Definition 2.1], which uses an idea of Goncharov [7]. We remark that $\zeta^{\mathfrak{m}}(2)$ is not treated to be zero in \mathcal{H} . Denote by \mathfrak{D} the depth filtration $\mathfrak{D}_r \mathcal{H} = \langle \zeta^{\mathfrak{m}}(\mathbf{n}) \mid \text{dep}(\mathbf{n}) \leq r \rangle_{\mathbb{Q}}$ (see [3, Section 4]), and by $\zeta^{\mathfrak{m}}_{\mathfrak{D}}(\mathbf{n})$ the depth-graded motivic MZV which is an image of $\zeta^{\mathfrak{m}}(\mathbf{n})$ in $\text{gr}^{\mathfrak{D}} \mathcal{H}$. Let $\mathcal{H}^{\text{odd}} \subset \mathcal{H}$ be the \mathbb{Q} -vector space generated by the elements

$$\zeta^{\mathfrak{m}}(n_1, \dots, n_r) \quad (r \geq 0, n_1, \dots, n_r \geq 3 : \text{odd}).$$

According to the result of Brown [3, Proposition 10.1], for each odd integer $m > 1$ one can define a well-defined derivation ∂_m such that

$$\partial_m : \text{gr}_r^{\mathfrak{D}} \mathcal{H}_N^{\text{odd}} \longrightarrow \text{gr}_{r-1}^{\mathfrak{D}} \mathcal{H}_{N-m}^{\text{odd}} \text{ and } \partial_m(\zeta^{\mathfrak{m}}_{\mathfrak{D}}(n)) = \delta \binom{n}{m},$$

where $\delta \binom{n}{m} = 1$ if $n = m$ and 0 otherwise. We sketch an explicit construction of the derivation ∂_m (we follow an idea used in [2, Section 5]).

Recall Brown's operator D_m (see [2, Definition 3.1]). This is an infinitesimal coaction obtained by the coaction of the algebra-comodule \mathcal{H} , and becomes a derivation (i.e. $D_m(\xi_1 \xi_2) = (1 \otimes \xi_1) D_m(\xi_2) + (1 \otimes \xi_2) D_m(\xi_1)$ for $\xi_1, \xi_2 \in \mathcal{H}$). The derivation D_m can be computed by using the following explicit formula: for odd $m > 1$

$$D_m(I^{\mathfrak{m}}(a_0; a_1, \dots, a_N; a_{N+1}))$$

$$= \sum_{p=0}^{N-n} I^{\mathfrak{L}}(a_p; a_{p+1}, \dots, a_{p+n}; a_{p+n+1}) \otimes I^{\mathfrak{m}}(a_0; a_1, \dots, a_p, a_{p+n+1}, \dots, a_N; a_{N+1}),$$

where for $a_i \in \{0, 1\}$ the element $I^{\mathfrak{m}}(a_0; a_1, \dots, a_N; a_{N+1}) \in \mathcal{H}_N$ is the motivic iterated integral (see [2, (2.16)]) and $I^{\mathfrak{L}}$ is an image of $I^{\mathfrak{m}}$ in $\mathcal{L} = \mathcal{A}_{>0}/(\mathcal{A}_{>0})^2$ where $\mathcal{A} = \mathcal{H}/\zeta^{\mathfrak{m}}(2)\mathcal{H}$. We remark that the motivic iterated integral defines the motivic multiple zeta value:

$$\zeta^{\mathfrak{m}}(n_1, \dots, n_r) = I^{\mathfrak{m}}(0; 1, \underbrace{0, \dots, 0}_{n_1-1}, \dots, 1, \underbrace{0, \dots, 0}_{n_r-1}, 1).$$

Let us denote by \mathcal{L}_m the weight m part of \mathcal{L} . From the result [3, Proposition 10.1] we have for odd $m > 1$

$$D_m(\mathfrak{D}_r \mathcal{H}_N^{\text{odd}}) \subset \mathcal{L}_m \otimes \mathfrak{D}_{r-1} \mathcal{H}_{N-m}^{\text{odd}} + \mathcal{L}_m \otimes \mathfrak{D}_{r-2} \mathcal{H}_{N-m},$$

which gives a map for $r \in \mathbb{Z}_{>0}$

$$\text{gr}_r^{\mathfrak{D}} D_m : \text{gr}_r^{\mathfrak{D}} \mathcal{H}_N^{\text{odd}} \longrightarrow \mathcal{L}_m \otimes_{\mathbb{Q}} \text{gr}_{r-1}^{\mathfrak{D}} \mathcal{H}_{N-m}^{\text{odd}}.$$

By computing $\text{gr}_r^{\mathfrak{D}} D_m(\zeta_{\mathfrak{D}}^{\mathfrak{m}}(\mathbf{n}))$ for $\mathbf{n} \in S_{N,r}$ (see an explicit formula in (4.4)), one can find

$$(2.2) \quad \text{gr}_r^{\mathfrak{D}} D_m(\text{gr}_r^{\mathfrak{D}} \mathcal{H}_N^{\text{odd}}) \subset \mathbb{Q} \zeta_m \otimes_{\mathbb{Q}} \text{gr}_{r-1}^{\mathfrak{D}} \mathcal{H}_{N-m}^{\text{odd}},$$

where ζ_m is an image of $\zeta^{\mathfrak{m}}(m)$ in \mathcal{L} . Then the above derivation ∂_m is defined to be $(\zeta_m^{\vee} \otimes 1) \circ \text{gr}_r^{\mathfrak{D}} D_m|_{\text{gr}_r^{\mathfrak{D}} \mathcal{H}_N^{\text{odd}}}$. Let us call $\zeta_{\mathfrak{D}}^{\mathfrak{m}}(\mathbf{n})$ the totally odd motivic MZVs when $\mathbf{n} \in S_{N,r}$. Using ∂_m , one can obtain a criterion for checking when a \mathbb{Q} -linear combination of totally odd motivic MZVs vanishes.

Proposition 2.2. *A \mathbb{Q} -linear combination ξ in $\mathfrak{D}_r \mathcal{H}_N^{\text{odd}}$ lies in $\mathfrak{D}_{r-1} \mathcal{H}_N$ if and only if it satisfies $\partial_{m_r} \circ \partial_{m_{r-1}} \circ \dots \circ \partial_{m_1}(\bar{\xi}) = 0$ for all $(m_1, \dots, m_r) \in S_{N,r}$, where $\bar{\xi}$ is an image of ξ in $\text{gr}_r^{\mathfrak{D}} \mathcal{H}_N^{\text{odd}}$.*

Proof. When $r = 2$, this immediately follows from [2, Theorem 3.3]. The reminder can be verified by induction on r together with (2.2). \square

We now define the matrix $C_{N,r}$. For totally odd indices $(n_1, \dots, n_r), (m_1, \dots, m_r) \in S_{N,r}$ we set

$$(2.3) \quad c_{\substack{m_1, \dots, m_r \\ n_1, \dots, n_r}}^{\substack{m_1, \dots, m_r \\ n_1, \dots, n_r}} = \partial_{m_r} \circ \partial_{m_{r-1}} \circ \dots \circ \partial_{m_1}(\zeta_{\mathfrak{D}}^{\mathfrak{m}}(n_1, \dots, n_r)).$$

We note that $c_{\substack{m \\ n}}^{\substack{m \\ n}} = \delta_{\substack{m \\ n}}^{\substack{m \\ n}}$.

Definition 2.3. For integers $N > r > 0$, we define the $|S_{N,r}| \times |S_{N,r}|$ matrix $C_{N,r}$ by

$$C_{N,r} = \left(c \binom{m_1, \dots, m_r}{n_1, \dots, n_r} \right)_{\substack{(m_1, \dots, m_r) \in S_{N,r} \\ (n_1, \dots, n_r) \in S_{N,r}}}.$$

As a consequence of Proposition 2.2, we find that the relation

$$\sum_{(n_1, \dots, n_r) \in S_{N,r}} a_{n_1, \dots, n_r} \zeta_{\mathfrak{D}}^{\mathfrak{m}}(n_1, \dots, n_r) = 0$$

holds if and only if $(a_{n_1, \dots, n_r})_{(n_1, \dots, n_r) \in S_{N,r}} \in \ker {}^t C_{N,r}$. Combining this fact with the period homomorphism $per : \mathcal{H} \rightarrow \mathbb{R}$ mapping the elements $\zeta^{\mathfrak{m}}(n_1, \dots, n_r)$ to the real number $\zeta(n_1, \dots, n_r)$, we obtain the precise statement of Proposition 1.1 as follows.

Proposition 2.4. The totally odd MZVs of weight N and depth r satisfy $\dim_{\mathbb{Q}} \ker {}^t C_{N,r}$ linear relations of the form

$$\sum_{(n_1, \dots, n_r) \in S_{N,r}} a_{n_1, \dots, n_r} \zeta(n_1, \dots, n_r) = 0$$

with $(a_{n_1, \dots, n_r})_{(n_1, \dots, n_r) \in S_{N,r}} \in \ker {}^t C_{N,r}$.

As a corollary, we have the following inequality.

Corollary 2.5. For $N > r > 0$ we have

$$\dim_{\mathbb{Q}} \mathcal{Z}_{N,r}^{\text{odd}} \leq \text{rank } C_{N,r} = |S_{N,r}| - \dim_{\mathbb{Q}} \ker C_{N,r}.$$

Remark. Brown [3] has computed $\text{rank } C_{N,r}$ up to $N = 30$, and suggested the following conjecture, which is called the ‘uneven’ part of motivic Broadhurst-Kreimer conjecture (see [3, Conjecture 5]): the generating function of the rank of the matrix $C_{N,r}$ is given by

$$(2.4) \quad 1 + \sum_{N > r > 0} \text{rank } C_{N,r} x^N y^r \stackrel{?}{=} \frac{1}{1 - \mathbb{O}(x)y + \mathbb{S}(x)y^2}.$$

3. LINEAR RELATION AMONG TOTALLY ODD MZVs AND EVEN PERIOD POLYNOMIALS

3.1. Polynomial representations of Ihara action. We recall the polynomial representation of the Ihara action of Brown [3, Section 6]. This provides an expression of the generating function of the integers $c \binom{m_1, \dots, m_r}{n_1, \dots, n_r}$.

A polynomial representation of the depth-graded version of the linearised Ihara action $\underline{\circ} : \mathbb{Q}[x_1, \dots, x_r] \otimes_{\mathbb{Q}} \mathbb{Q}[x_1, \dots, x_s] \rightarrow \mathbb{Q}[x_1, \dots, x_{r+s}]$ is given explicitly by

$$(3.1) \quad \begin{aligned} f \underline{\circ} g(x_1, \dots, x_{r+s}) &= \sum_{i=0}^s f(x_{i+1} - x_i, \dots, x_{i+r} - x_i) g(x_1, \dots, \hat{x}_{i+1}, \dots, x_{r+s}) \\ &+ (-1)^{\deg(f)+r} \sum_{i=1}^s f(x_{i+r-1} - x_{i+r}, \dots, x_i - x_{i+r}) g(x_1, \dots, \hat{x}_i, \dots, x_{r+s}) \end{aligned}$$

for homogeneous polynomials $f(x_1, \dots, x_r)$ and $g(x_1, \dots, x_s)$, where $x_0 = 0$. We note that, by duality, for totally odd indices $(m_1, \dots, m_r), (n_1, \dots, n_r) \in S_{N,r}$ the integer $c_{n_1, \dots, n_r}^{(m_1, \dots, m_r)}$ defined in (2.3) coincides with the coefficient of $x_1^{n_1-1} \dots x_r^{n_r-1}$ in $x_1^{m_1-1} \underline{\circ} (\dots x_1^{m_{r-2}-1} \underline{\circ} (x_1^{m_{r-1}-1} \underline{\circ} x_1^{m_r-1}) \dots)$, i.e.

$$(3.2) \quad \begin{aligned} x_1^{m_1-1} \underline{\circ} (\dots x_1^{m_{r-2}-1} \underline{\circ} (x_1^{m_{r-1}-1} \underline{\circ} x_1^{m_r-1}) \dots) \\ = \sum_{\substack{n_1 + \dots + n_r = m_1 + \dots + m_r \\ n_1, \dots, n_r \geq 1}} c_{n_1, \dots, n_r}^{(m_1, \dots, m_r)} x_1^{n_1-1} \dots x_r^{n_r-1}. \end{aligned}$$

For integers $m_1, \dots, m_r, n_1, \dots, n_r \geq 1$, let $e_{n_1, \dots, n_r}^{(m_1, \dots, m_r)}$ be the integer obtained from the coefficient of $x_1^{n_1-1} \dots x_r^{n_r-1}$ in $x_1^{m_1-1} \underline{\circ} (x_1^{m_2-1} \dots x_{r-1}^{m_r-1})$:

$$(3.3) \quad x_1^{m_1-1} \underline{\circ} (x_1^{m_2-1} \dots x_{r-1}^{m_r-1}) = \sum_{\substack{n_1 + \dots + n_r = m_1 + \dots + m_r \\ n_1, \dots, n_r \geq 1}} e_{n_1, \dots, n_r}^{(m_1, \dots, m_r)} x_1^{n_1-1} \dots x_r^{n_r-1},$$

and $e_{n_1}^{(m_1)} = \delta_{n_1}^{(m_1)}$. For the latter purpose we now give explicit formulas of the integers $e_{n_1, \dots, n_r}^{(m_1, \dots, m_r)}$. For integers $m_1, \dots, m_r, n_1, \dots, n_r, m, n, n' \geq 1$ let us define $\delta_{n_1, \dots, n_r}^{(m_1, \dots, m_r)}$ as the Kronecker delta given by

$$\delta_{n_1, \dots, n_r}^{(m_1, \dots, m_r)} = \begin{cases} 1 & \text{if } m_i = n_i \text{ for all } i \in \{1, \dots, r\} \\ 0 & \text{otherwise} \end{cases},$$

and the integer $b_{n, n'}^m$ by

$$b_{n, n'}^m = (-1)^n \binom{m-1}{n-1} + (-1)^{n'-m} \binom{m-1}{n'-1}.$$

It is obvious that for odd $n, n', m > 1$ one has

$$(3.4) \quad b_{n, n'}^m + b_{n', n}^m = 0.$$

Lemma 3.1. *For integers $m_1, \dots, m_r, n_1, \dots, n_r \geq 1$, we have*

$$e\left(\begin{smallmatrix} m_1, \dots, m_r \\ n_1, \dots, n_r \end{smallmatrix}\right) = \delta\left(\begin{smallmatrix} m_1, \dots, m_r \\ n_1, \dots, n_r \end{smallmatrix}\right) + \sum_{i=1}^{r-1} \delta\left(\begin{smallmatrix} m_2, \dots, m_i, m_{i+2}, \dots, m_r \\ n_1, \dots, n_{i-1}, n_{i+2}, \dots, n_r \end{smallmatrix}\right) b_{n_i, n_{i+1}}^{m_1}.$$

Proof. By definition (3.1), one can compute

$$\begin{aligned} x_1^{m_1-1} \underline{\circ} (x_1^{m_2-1} \cdots x_{r-1}^{m_r-1}) &= x_1^{m_1-1} \cdots x_r^{m_r-1} \\ &+ \sum_{i=1}^{r-1} (x_{i+1} - x_i)^{m_1-1} (x_1^{m_2-1} \cdots x_i^{m_{i+1}-1} x_{i+2}^{m_{i+2}-1} \cdots x_r^{m_r-1} \\ &- x_1^{m_2-1} \cdots x_{i-1}^{m_i-1} x_{i+1}^{m_{i+1}-1} \cdots x_r^{m_r-1}) \\ &= x_1^{m_1-1} \cdots x_r^{m_r-1} + \sum_{i=1}^{r-1} x_1^{m_2-1} \cdots x_{i-1}^{m_i-1} x_{i+2}^{m_{i+2}-1} \cdots x_r^{m_r-1} \\ &\times \sum_{\substack{n_i+n_{i+1}=m_1+m_{i+1} \\ n_i, n_{i+1} \geq 1}} \left((-1)^{m_1-n_{i+1}} \binom{m_1-1}{n_{i+1}-1} - (-1)^{n_i-1} \binom{m_1-1}{n_i-1} \right) x_i^{n_i-1} x_{i+1}^{n_{i+1}-1} \\ &= \sum_{\substack{n_1+\dots+n_r=m_1+\dots+m_r \\ n_i \geq 1}} \left(\delta\left(\begin{smallmatrix} m_1, \dots, m_r \\ n_1, \dots, n_r \end{smallmatrix}\right) + \sum_{i=1}^{r-1} \delta\left(\begin{smallmatrix} m_2, \dots, m_i, m_{i+2}, \dots, m_r \\ n_1, \dots, n_{i-1}, n_{i+2}, \dots, n_r \end{smallmatrix}\right) b_{n_i, n_{i+1}}^{m_1} \right) x_1^{n_1-1} \cdots x_r^{n_r-1}, \end{aligned}$$

which completes the proof. \square

Remark. One interesting thing about the integer $e\left(\begin{smallmatrix} m_1, \dots, m_r \\ n_1, \dots, n_r \end{smallmatrix}\right)$ is that it can be obtained as a part of coefficients of the Fourier expansion of the multiple Eisenstein series. In general, the author and Bachmann [1] showed the correspondence between the Fourier expansion of multiple Eisenstein series and the coproduct Δ defined by Goncharov [7]. However, we were not able to prove any linear relations among totally odd MZVs from this correspondence whenever $r \geq 3$.

We end this section by defining the matrix $E_{N,r}$ we are actually interested in.

Definition 3.2. *For $N > r > 0$, we define the $|S_{N,r}| \times |S_{N,r}|$ matrix $E_{N,r}$ by*

$$E_{N,r} = \left(e\left(\begin{smallmatrix} m_1, \dots, m_r \\ n_1, \dots, n_r \end{smallmatrix}\right) \right)_{\substack{(m_1, \dots, m_r) \in S_{N,r} \\ (n_1, \dots, n_r) \in S_{N,r}}}.$$

3.2. The relation between $E_{N,r}$ and $C_{N,r}$. We now prove Proposition 1.2.

For integers $r \geq 2$ and $2 \leq q \leq r$, let $E_{N,r}^{(q)}$ be the $|S_{N,r}| \times |S_{N,r}|$ matrix defined by

$$(3.5) \quad E_{N,r}^{(q)} = \left(\delta \binom{m_1, \dots, m_{r-q}}{n_1, \dots, n_{r-q}} \cdot e \binom{m_{r-q+1}, \dots, m_r}{n_{r-q+1}, \dots, n_r} \right)_{\substack{(m_1, \dots, m_r) \in S_{N,r} \\ (n_1, \dots, n_r) \in S_{N,r}}}.$$

We note $E_{N,r}^{(r)} = E_{N,r}$. The following gives a connection between $E_{N,r}$ and $C_{N,r}$ (this was indicated by Seidai Yasuda), from which Proposition 1.2 immediately follows.

Proposition 3.3. *For integers $N > r > 1$, we have*

$$C_{N,r} = E_{N,r}^{(2)} \cdot E_{N,r}^{(3)} \cdots E_{N,r}^{(r-1)} \cdot E_{N,r}.$$

Proof. This is shown by induction on r . When $r = 2$, the assertion $C_{N,2} = E_{N,2}$ follows from the definition of the integer $e \binom{m_1, \dots, m_r}{n_1, \dots, n_r}$ in (3.3) and the generating function of $c \binom{m_1, \dots, m_r}{n_1, \dots, n_r}$ in (3.2). For $r \geq 3$, we set

$$\begin{aligned} f(x_1, \dots, x_{r-1}) &= x_1^{m_2-1} \circ (\cdots \circ (x_1^{m_{r-1}-1} \circ x_1^{m_r-1}) \cdots) \\ &= \sum_{\substack{n_2 + \cdots + n_r = m_2 + \cdots + m_r \\ n_2, \dots, n_r \geq 1}} c \binom{m_2, \dots, m_r}{n_2, \dots, n_r} x_1^{n_2-1} \cdots x_{r-1}^{n_r-1}. \end{aligned}$$

By definition, we see that $x_1^{m_1-1} \circ f(x_1, \dots, x_{r-1}) = \sum_{n_1 + \cdots + n_r = N} c \binom{m_1, \dots, m_r}{n_1, \dots, n_r} x_1^{n_1-1} \cdots x_r^{n_r-1}$, where $N = m_1 + \cdots + m_r$. One can also compute by linearity

$$\begin{aligned} x_1^{m_1-1} \circ f(x_1, \dots, x_{r-1}) &= \sum_{t_2 + \cdots + t_r = m_2 + \cdots + m_r} c \binom{m_2, \dots, m_r}{t_2, \dots, t_r} x_1^{m_1-1} \circ (x_1^{t_2-1} \cdots x_{r-1}^{t_r-1}) \\ &= \sum_{t_1 + \cdots + t_r = N} \delta \binom{m_1}{t_1} c \binom{m_2, \dots, m_r}{t_2, \dots, t_r} x_1^{t_1-1} \circ (x_1^{t_2-1} \cdots x_{r-1}^{t_r-1}) \\ &= \sum_{t_1 + \cdots + t_r = N} \delta \binom{m_1}{t_1} c \binom{m_2, \dots, m_r}{t_2, \dots, t_r} \sum_{n_1 + \cdots + n_r = N} e \binom{t_1, \dots, t_r}{n_1, \dots, n_r} x_1^{n_1-1} \cdots x_r^{n_r-1} \\ &= \sum_{n_1 + \cdots + n_r = N} \left(\sum_{t_1 + \cdots + t_r = N} \delta \binom{m_1}{t_1} c \binom{m_2, \dots, m_r}{t_2, \dots, t_r} e \binom{t_1, \dots, t_r}{n_1, \dots, n_r} \right) x_1^{n_1-1} \cdots x_r^{n_r-1}, \end{aligned}$$

where $t_1, \dots, t_r \geq 1$. For the totally odd indices $(m_1, \dots, m_r), (n_1, \dots, n_r) \in S_{N,r}$, the term $\delta \binom{m_1}{t_1} e \binom{t_1, \dots, t_r}{n_1, \dots, n_r}$ in the last equation is 0 if $(t_1, \dots, t_r) \notin S_{N,r}$ with $t_1 + \cdots + t_r = N$ (this follows from the explicit formula of $e \binom{m_1, \dots, m_r}{n_1, \dots, n_r}$ in Lemma 3.1), so one has

$$c \binom{m_1, \dots, m_r}{n_1, \dots, n_r} = \sum_{(t_1, \dots, t_r) \in S_{N,r}} \delta \binom{m_1}{t_1} c \binom{m_2, \dots, m_r}{t_2, \dots, t_r} e \binom{t_1, \dots, t_r}{n_1, \dots, n_r}.$$

Then our claim follows from the induction hypothesis

$$\left(\delta \binom{m_1}{n_1} c \binom{m_2, \dots, m_r}{n_2, \dots, n_r} \right)_{\substack{(m_1, \dots, m_r) \in S_{N,r} \\ (n_1, \dots, n_r) \in S_{N,r}}} = E_{N,r}^{(2)} \cdot E_{N,r}^{(3)} \cdots E_{N,r}^{(r-1)}.$$

□

By Propositions 2.4 and 3.3, one can obtain linear relations among totally odd MZVs from $\ker {}^t E_{N,r}$. Let us illustrate a few examples of these relations. For the matrix $E_{N,2}$, the first example of relations is obtained from the matrix

$$E_{12,2} = \begin{pmatrix} e \binom{3,9}{3,9} & e \binom{3,9}{5,7} & e \binom{3,9}{7,5} & e \binom{3,9}{9,3} \\ e \binom{5,7}{3,9} & e \binom{5,7}{5,7} & e \binom{5,7}{7,5} & e \binom{5,7}{9,3} \\ e \binom{7,5}{5,9} & e \binom{7,5}{5,7} & e \binom{7,5}{7,5} & e \binom{7,5}{9,3} \\ e \binom{9,3}{3,9} & e \binom{9,3}{5,7} & e \binom{9,3}{7,5} & e \binom{9,3}{9,3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -6 & 0 & 1 & 6 \\ -15 & -14 & 15 & 15 \\ -27 & -42 & 42 & 28 \end{pmatrix}.$$

The space $\ker {}^t E_{12,2}$ is generated by the vector $(14, 75, 84, 0)$. This gives the relation

$$14\zeta_{\mathfrak{D}}(3, 9) + 75\zeta_{\mathfrak{D}}(5, 7) + 84\zeta_{\mathfrak{D}}(7, 5) = 0.$$

In the case of $r = 3$, we consider the matrix

$$E_{15,3} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -6 & -6 & 1 & 6 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & -6 & 1 & 6 & 6 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 & 0 & 0 & -5 & 6 & 6 \\ -15 & -14 & 0 & 15 & 0 & 0 & 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & -15 & -14 & 0 & 0 & 0 & 15 & 15 \\ -27 & -42 & 42 & 0 & 0 & 0 & -42 & 0 & 42 & 28 \end{pmatrix}.$$

The space $\ker {}^t E_{15,3}$ is generated by $(-14, 15, 6, 0, 0, 36, 0, 0, 0, 0)$, which gives

$$-14\zeta_{\mathfrak{D}}(3, 3, 9) + 15\zeta_{\mathfrak{D}}(3, 5, 7) + 6\zeta_{\mathfrak{D}}(3, 7, 5) + 36\zeta_{\mathfrak{D}}(5, 5, 5) = 0.$$

The following relation is obtained from the space $\ker {}^t E_{18,4}$:

$$70\zeta_{\mathfrak{D}}(3, 3, 3, 9) - 75\zeta_{\mathfrak{D}}(3, 3, 5, 7) - 30\zeta_{\mathfrak{D}}(3, 3, 7, 5) + 36\zeta_{\mathfrak{D}}(3, 5, 5, 5) = 0.$$

3.3. Relation with period polynomials. This section is devoted to proving Theorem 1.3.

Let $\mathbf{P}_{N,r} \subset \mathbb{Q}[x_1, \dots, x_r]$ be the $|S_{N,r}|$ -dimensional \mathbb{Q} -vector space spanned by the set $\{x_1^{n_1-1} \cdots x_r^{n_r-1} \mid (n_1, \dots, n_r) \in S_{N,r}\}$, and $\mathbf{W}_{N,r}$ its subspace defined for $r \geq 2$ by $\mathbf{W}_{N,r} = \{p \in \mathbf{P}_{N,r} \mid p(x_1, \dots, x_r) = p(x_2 - x_1, x_2, \underbrace{x_3, \dots, x_r}_{r-2}) - p(x_2 - x_1, x_1, \underbrace{x_3, \dots, x_r}_{r-2})\}$.

The space $\mathbf{W}_{N,2}$ is called the space of restricted even period polynomials (see [4]). As mentioned in the Introduction, the dimension of the space $\mathbf{W}_{N,2}$ is equal to the dimension of the \mathbb{C} -vector space of cusp forms of weight N for Γ_1 , so that we have

$$(3.6) \quad \sum_{N>0} \dim_{\mathbb{Q}} \mathbf{W}_{N,2} x^N = \mathbb{S}(x).$$

For $r \geq 3$, from definition we easily find that $\mathbf{W}_{N,r} \cong \bigoplus_{1 < n < N} \mathbf{W}_{n,2} \otimes_{\mathbb{Q}} \mathbf{P}_{N-n,r-2}$. In fact, every element in $\mathbf{W}_{N,r}$ can be written as \mathbb{Q} -linear combinations of the form $p(x_1, x_2) x_3^{n_3-1} \cdots x_r^{n_r-1}$ ($p(x_1, x_2) \in \mathbf{W}_{n,2}$, $(n_3, \dots, n_r) \in S_{N-n,r-2}$, $1 < n < N$). Thus, from $\mathbb{O}(x)^r = \sum_{N>0} |S_{N,r}| x^N$ and (3.6), we have an exact dimension of the space $\mathbf{W}_{N,r}$:

$$(3.7) \quad \sum_{N>0} \dim_{\mathbb{Q}} \mathbf{W}_{N,r} x^N = \mathbb{S}(x) \cdot \mathbb{O}(x)^{r-2}.$$

Baumard and Schneps [4] have shown that the space $\mathbf{W}_{N,2}$ is isomorphic to the space $\ker E_{N,2}$. Since this fact is used for proving Theorem 1.4, we reprove it for completeness. Set an isomorphism $\pi_1 (= \pi_1^{(N,r)})$ as follows.

$$\begin{aligned} \pi_1 : \mathbf{P}_{N,r} &\longrightarrow \mathbf{Vect}_{N,r} \\ \sum_{(n_1, \dots, n_r) \in S_{N,r}} a_{n_1, \dots, n_r} x_1^{n_1-1} \cdots x_r^{n_r-1} &\longmapsto (a_{n_1, \dots, n_r})_{(n_1, \dots, n_r) \in S_{N,r}}. \end{aligned}$$

Proposition 3.4. ([4, Proposition 3.2]) *For each integer $N > 0$ one has*

$$\pi_1(\mathbf{W}_{N,2}) = \ker E_{N,2}.$$

Proof. For every polynomial $p(x_1, x_2) = \sum_{(n_1, n_2) \in S_{N,2}} a_{n_1, n_2} x_1^{n_1-1} x_2^{n_2-1}$ satisfying $p(x, x) = 0$, one can compute

$$p(x_1, x_2) - p(x_2 - x_1, x_2) + p(x_2 - x_1, x_1)$$

$$\begin{aligned}
 &= \sum_{(m_1, m_2) \in S_{N,2}} a_{m_1, m_2} \sum_{\substack{n_1 + n_2 = N \\ n_1, n_2 \geq 2}} \left(\delta \binom{m_1, m_2}{n_1, n_2} - (-1)^{n_1-1} \binom{m_1-1}{n_1-1} \right. \\
 &\quad \left. + (-1)^{m_1-n_2} \binom{m_1-1}{n_2-1} \right) x_1^{n_1-1} x_2^{n_2-1} \\
 &= \sum_{\substack{n_1 + n_2 = N \\ n_1, n_2 \geq 2}} \left(\sum_{(m_1, m_2) \in S_{N,2}} a_{m_1, m_2} e \binom{m_1, m_2}{n_1, n_2} \right) x_1^{n_1-1} x_2^{n_2-1} \\
 (3.8) \quad &= \sum_{(n_1, n_2) \in S_{N,2}} \left(\sum_{(m_1, m_2) \in S_{N,2}} a_{m_1, m_2} e \binom{m_1, m_2}{n_1, n_2} \right) x_1^{n_1-1} x_2^{n_2-1} \\
 (3.9) \quad &+ \frac{1}{2} (p(x_2 - x_1, x_1) - p(x_2 - x_1, x_2) - p(x_2 + x_1, x_1) + p(x_2 + x_1, x_2)).
 \end{aligned}$$

Assume $p(x_1, x_2) \in \mathbf{W}_{N,2}$. Since $p(x_1, x_2) = p(x_2 - x_1, x_2) - p(x_2 - x_1, x_1) = p(x_2 + x_1, x_2) - p(x_2 + x_1, x_1)$, the polynomial in (3.9) is zero. Then the polynomial in (3.8) has to be 0, which implies $\pi_1(p) \in \ker E_{N,2}$. We now prove $\pi_1^{-1}(v)(x_1, x_2) \in \mathbf{W}_{N,2}$ for $v \in \ker E_{N,2}$ using the action of the group $\mathrm{PGL}_2(\mathbb{Z})$ on $\mathbb{Q}[x_1, x_2]_{(N-2)}$ (N : even) defined by $(f|\gamma)(x_1, x_2) = f(ax_1 + bx_2, cx_1 + dx_2)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Set

$$\delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Z}).$$

For $v = (a_{n_1, n_2})_{(n_1, n_2) \in S_{N,2}} \in \ker E_{N,2}$, one has

$$0 = E_{N,2}(v) = \left(a_{n_1, n_2} + \sum_{(m_1, m_2) \in S_{N,2}} a_{m_1, m_2} b_{n_1, n_2}^{m_1} \right)_{(n_1, n_2) \in S_{N,2}}.$$

By (3.4), one obtains $a_{n_1, n_2} = -a_{n_2, n_1}$. This shows $\pi_1^{-1}(v)|(\varepsilon + 1) = 0$, where we have extended the action of $\mathrm{PGL}_2(\mathbb{Z})$ to its group ring by linearity. We notice that $\pi_1^{-1}(v)|(\delta - 1) = 0$ because of even. Using $T\delta = \delta T^{-1}$ and $T\varepsilon\delta = \varepsilon T\varepsilon T^{-1}$, we have $\pi_1^{-1}(v)|(1 - T + T\varepsilon)\delta = \pi_1^{-1}(v)|(1 - T^{-1} + \varepsilon T\varepsilon T^{-1}) = -\pi_1^{-1}(v)|(1 - T + T\varepsilon)T^{-1}$. Let $G = \pi_1^{-1}(v)|(1 - T + T\varepsilon)$. Then $G(0, x_2) = 0$ and

$$0 = 2 \times (3.8) = G|(1 + \delta) = G|(1 - T^{-1}).$$

Since the coefficient matrix obtained by the action of $1 - T^{-1}$ on G becomes triangular, one can obtain $G = 0$. The assertion follows from

$$\begin{aligned}
 0 &= G(x_1, x_2) = \pi_1^{-1}(v)(x_1, x_2) - \pi_1^{-1}(v)(x_2 + x_1, x_2) + \pi_1^{-1}(v)(x_2 + x_1, x_1) \\
 &= \pi_1^{-1}(v)(x_1, x_2) - \pi_1^{-1}(v)(x_2 - x_1, x_2) + \pi_1^{-1}(v)(x_2 - x_1, x_1).
 \end{aligned}$$

□

From (3.6) one immediately obtains an exact value of $\text{rank } E_{N,2}$.

Corollary 3.5. *The generating function of the rank of the matrix $E_{N,2}$ is given by*

$$\sum_{N>0} \text{rank } E_{N,2} x^N = \mathbb{O}(x)^2 - \mathbb{S}(x),$$

or equivalently, one has $\sum_{N>0} \dim_{\mathbb{Q}} \ker E_{N,2} x^N = \mathbb{S}(x)$.

We now present our result for $r \geq 3$. Set the identity map $I_{N,r} \in M_{|\mathbb{S}_{N,r}|}(\mathbb{Z})$ on the vector space $\mathbf{Vect}_{N,r}$:

$$I_{N,r} = \left(\delta \binom{m_1, \dots, m_r}{n_1, \dots, n_r} \right)_{\substack{(m_1, \dots, m_r) \in \mathbb{S}_{N,r} \\ (n_1, \dots, n_r) \in \mathbb{S}_{N,r}}}.$$

Theorem 3.6. *Let r be a positive integer greater than 2 and $F_{N,r}$ the matrix $E_{N,r} - I_{N,r}$. Then, the following \mathbb{Q} -linear map is injective:*

$$\begin{aligned} \mathbf{W}_{N,r} &\longrightarrow \ker E_{N,r} \\ p(x_1, \dots, x_r) &\longmapsto F_{N,r}(\pi_1(p(x_1, \dots, x_r))). \end{aligned}$$

Proof. We first check that $F_{N,r}(\pi_1(p)) \in \ker E_{N,r}$ for any polynomial $p \in \mathbf{W}_{N,r}$. Define the action $\sigma_r^{(i)}$ for a polynomial $f(x_1, \dots, x_r)$ and $i \in \{1, 2, \dots, r-1\}$ by

$$f(x_1, \dots, x_r) \big| \sigma_r^{(i)} = f(x_{i+1} - x_i, x_1, \dots, \hat{x}_{i+1}, \dots, x_r) - f(x_{i+1} - x_i, x_1, \dots, \hat{x}_i, \dots, x_r).$$

Note that any polynomial p in $\mathbf{W}_{N,r}$ satisfies $p \big| (1 + \sigma_r^{(1)}) = 0$ (by definition), and then we have

$$(3.10) \quad p(x_1, \dots, x_r) + p(x_2, x_1, x_3, \dots, x_r) = 0.$$

We now prove for $p \in \mathbf{W}_{N,r}$

$$(3.11) \quad p(x_1, \dots, x_r) \big| (\sigma_r^{(j)} \sigma_r^{(i)} + \sigma_r^{(i)} \sigma_r^{(j-1)}) = 0 \quad (r-1 \geq i \geq j \geq 2),$$

where $p \big| (\sigma_r^{(j)} \sigma_r^{(i)} + \sigma_r^{(i)} \sigma_r^{(j-1)})$ means $(p \big| \sigma_r^{(j)}) \big| \sigma_r^{(i)} + (p \big| \sigma_r^{(i)}) \big| \sigma_r^{(j-1)}$. For $r-1 \geq i > j \geq 2$ one computes

$$(3.12) \quad \begin{aligned} &p(x_1, \dots, x_r) \big| \sigma_r^{(j)} \sigma_r^{(i)} \\ &= p(x_j - x_{j-1}, x_{i+1} - x_i, x_1, \dots, \hat{x}_j, \dots, \hat{x}_{i+1}, \dots, x_r) \end{aligned}$$

$$(3.13) \quad - p(x_j - x_{j-1}, x_{i+1} - x_i, x_1, \dots, \hat{x}_{j-1}, \dots, \hat{x}_{i+1}, \dots, x_r)$$

$$(3.14) \quad - p(x_j - x_{j-1}, x_{i+1} - x_i, x_1, \dots, \hat{x}_j, \dots, \hat{x}_i, \dots, x_r)$$

$$(3.15) \quad + p(x_j - x_{j-1}, x_{i+1} - x_i, x_1, \dots, \hat{x}_{j-1}, \dots, \hat{x}_i, \dots, x_r),$$

and

$$(3.16) \quad p(x_1, \dots, x_r) \Big| \sigma_r^{(i)} \sigma_r^{(j-1)} = p(x_{i+1} - x_i, x_j - x_{j-1}, x_1, \dots, \hat{x}_j, \dots, \hat{x}_{i+1}, \dots, x_r)$$

$$(3.17) \quad - p(x_{i+1} - x_i, x_j - x_{j-1}, x_1, \dots, \hat{x}_j, \dots, \hat{x}_i, \dots, x_r)$$

$$(3.18) \quad - p(x_{i+1} - x_i, x_j - x_{j-1}, x_1, \dots, \hat{x}_{j-1}, \dots, \hat{x}_{i+1}, \dots, x_r)$$

$$(3.19) \quad + p(x_{i+1} - x_i, x_j - x_{j-1}, x_1, \dots, \hat{x}_{j-1}, \dots, \hat{x}_i, \dots, x_r).$$

From (3.10) it follows $(3.12) + (3.16) = (3.13) + (3.18) = (3.14) + (3.17) = (3.15) + (3.19) = 0$. Thus we have (3.11) for $r - 1 \geq i > j \geq 2$. When $j = i$, one can check

$$(3.20) \quad p(x_1, \dots, x_r) \Big| \sigma_r^{(j)} \sigma_r^{(j)} = p(x_j - x_{j-1}, x_{j+1} - x_j, x_1, \dots, \hat{x}_j, \hat{x}_{j+1}, \dots, x_r)$$

$$(3.21) \quad - p(x_j - x_{j-1}, x_{j+1} - x_j, x_1, \dots, \hat{x}_{j-1}, x_j, \hat{x}_{j+1}, \dots, x_r)$$

$$(3.22) \quad - p(x_{j+1} - x_{j-1}, x_{j+1} - x_j, x_1, \dots, \hat{x}_j, \hat{x}_{j+1}, \dots, x_r)$$

$$(3.23) \quad + p(x_{j+1} - x_{j-1}, x_{j+1} - x_j, x_1, \dots, \hat{x}_{j-1}, \hat{x}_j, \dots, x_r),$$

and

$$(3.24) \quad p(x_1, \dots, x_r) \Big| \sigma_r^{(j)} \sigma_r^{(j-1)} = p(x_{j+1} - x_{j-1}, x_j - x_{j-1}, x_1, \dots, \hat{x}_j, \hat{x}_{j+1}, \dots, x_r)$$

$$(3.25) \quad - p(x_{j+1} - x_{j-1}, x_j - x_{j-1}, x_1, \dots, \hat{x}_{j-1}, \hat{x}_j, \dots, x_r)$$

$$(3.26) \quad - p(x_{j+1} - x_j, x_j - x_{j-1}, x_1, \dots, \hat{x}_{j-1}, x_j, \hat{x}_{j+1}, \dots, x_r)$$

$$(3.27) \quad + p(x_{j+1} - x_j, x_j - x_{j-1}, x_1, \dots, \hat{x}_{j-1}, \hat{x}_j, \dots, x_r).$$

From (3.10), it follows $(3.21) + (3.26) = 0$. For the relation $p|(1 + \sigma_r^{(1)}) = 0$, the substitutions $x_1 \rightarrow x_j - x_{j-1}, x_2 \rightarrow x_{j+1} - x_j, x_3 \rightarrow x_1, \dots, x_{j+1} \rightarrow x_{j-1}$ shows $(3.20) + (3.22) + (3.24) = 0$, and the substitutions $x_1 \rightarrow x_{j+1} - x_{j-1}, x_2 \rightarrow x_{j+1} - x_j, x_3 \rightarrow x_1, \dots, x_j \rightarrow x_{j-2}$ together with (3.10) lead to $(3.23) + (3.25) + (3.27) = 0$.

Thus we have (3.11). From this, putting $\sigma_r = \sigma_r^{(1)} + \cdots + \sigma_r^{(r-1)}$ one can obtain

$$(3.28) \quad \begin{aligned} p|\sigma_r|(1 + \sigma_r) &= p|(\sigma_r^{(2)} + \cdots + \sigma_r^{(r-1)})|\sigma_r \\ &= \sum_{r-1 \geq i \geq j \geq 2} p|(\sigma_r^{(j)}\sigma_r^{(i)} + \sigma_r^{(i)}\sigma_r^{(j-1)})| = 0, \end{aligned}$$

where for the first equality we have used $p|(1 + \sigma_r^{(1)})| = 0$. We now compute the coefficient of $x_1^{n_1-1} \cdots x_r^{n_r-1}$ in $p|\sigma_r|(1 + \sigma_r)$. Notice that by (3.1) one has $x_1^{m_1-1} \cdots x_r^{m_r-1}|(1 + \sigma_r) = x_1^{m_1-1} \sqcup (x_1^{m_2-1} \cdots x_{r-1}^{m_{r-1}-1})$ (see also the proof of Lemma 3.1). Thus for every polynomial $p(x_1, \dots, x_r) = \sum_{(n_1, \dots, n_r) \in S_{N,r}} a_{n_1, \dots, n_r} x_1^{n_1-1} \cdots x_r^{n_r-1}$ one can compute

$$\begin{aligned} &p|\sigma_r|(1 + \sigma_r) \\ &= \sum_{(m_1, \dots, m_r) \in S_{N,r}} a_{m_1, \dots, m_r} \sum_{t_1 + \cdots + t_r = N} (e_{t_1, \dots, t_r}^{(m_1, \dots, m_r)} - \delta_{t_1, \dots, t_r}^{(m_1, \dots, m_r)}) x_1^{t_1-1} \cdots x_r^{t_r-1} |(1 + \sigma_r) \\ &= \sum_{n_1 + \cdots + n_r = N} \sum_{(m_1, \dots, m_r) \in S_{N,r}} a_{m_1, \dots, m_r} \\ &\quad \times \left(\sum_{t_1 + \cdots + t_r = N} (e_{t_1, \dots, t_r}^{(m_1, \dots, m_r)} - \delta_{t_1, \dots, t_r}^{(m_1, \dots, m_r)}) e_{n_1, \dots, n_r}^{(t_1, \dots, t_r)} \right) x_1^{n_1-1} \cdots x_r^{n_r-1}. \end{aligned}$$

Recalling the computation in the proof of Lemma 3.1, we have known $x_1^{m_1-1} \cdots x_r^{m_r-1} |\sigma_r^{(i)} = \sum_{n_1 + \cdots + n_r = N} \delta_{n_1, \dots, n_{i-1}, n_{i+2}, \dots, n_r}^{(m_2, \dots, m_i, m_{i+2}, \dots, m_r)} b_{n_i, n_{i+1}}^{m_1} x_1^{n_1-1} \cdots x_r^{n_r-1}$, where $N = m_1 + \cdots + m_r$, and hence for $1 \leq i, j \leq r-1$

$$\begin{aligned} &x_1^{m_1-1} \cdots x_r^{m_r-1} |\sigma_r^{(j)}| \sigma_r^{(i)} \\ &= \sum_{n_1 + \cdots + n_r = N} \left(\sum_{t_1 + \cdots + t_r = N} \delta_{t_1, \dots, t_{j-1}, t_{j+2}, \dots, t_r}^{(m_2, \dots, m_j, m_{j+2}, \dots, m_r)} b_{t_j, t_{j+1}}^{m_1} \right. \\ &\quad \left. \times \delta_{n_1, \dots, n_{i-1}, n_{i+2}, \dots, n_r}^{(t_2, \dots, t_i, t_{i+2}, \dots, t_r)} b_{n_i, n_{i+1}}^{t_1} \right) x_1^{n_1-1} \cdots x_r^{n_r-1}. \end{aligned}$$

Here $\delta_{t_1, \dots, t_{j-1}, t_{j+2}, \dots, t_r}^{(m_2, \dots, m_j, m_{j+2}, \dots, m_r)} \delta_{n_1, \dots, n_{i-1}, n_{i+2}, \dots, n_r}^{(t_2, \dots, t_i, t_{i+2}, \dots, t_r)}$ is 0 if $(t_1, \dots, t_r) \notin S_{N,r}$ and $(m_1, \dots, m_r), (n_1, \dots, n_r) \in S_{N,r}$. This shows that for $(n_1, \dots, n_r) \in S_{N,r}$ the coefficient of $x_1^{n_1-1} \cdots x_r^{n_r-1}$ in $p|\sigma_r|(1 + \sigma_r)$ is given by

$$\sum_{(m_1, \dots, m_r) \in S_{N,r}} a_{m_1, \dots, m_r} \left(\sum_{(t_1, \dots, t_r) \in S_{N,r}} (e_{t_1, \dots, t_r}^{(m_1, \dots, m_r)} - \delta_{t_1, \dots, t_r}^{(m_1, \dots, m_r)}) e_{n_1, \dots, n_r}^{(t_1, \dots, t_r)} \right),$$

which is exactly the (n_1, \dots, n_r) -th entry of the row vector $E_{N,r}(F_{N,r}(\pi_1(p)))$. As a result, (3.28) implies $E_{N,r}(F_{N,r}(\pi_1(p))) = 0$.

Finally, we prove the injectivity of the map $F_{N,r} \circ \pi_1$. It suffices to check that

$$\pi_1(\mathbf{W}_{N,r}) \cap \ker F_{N,r} = \{0\}.$$

Set $\pi_1(p) = (a_{n_1, \dots, n_r})_{(n_1, \dots, n_r) \in S_{N,r}}$ for $p \in \mathbf{W}_{N,r}$. For each $(n_1, \dots, n_r) \in S_{N,r}$, the assumption $p \in \mathbf{W}_{N,r}$ gives the relation

$$\sum_{(m_1, \dots, m_r) \in S_{N,r}} \left(\delta \binom{m_1, \dots, m_r}{n_1, \dots, n_r} + \delta \binom{m_3, \dots, m_r}{n_3, \dots, n_r} b_{n_1, n_2}^{m_1} \right) a_{m_1, \dots, m_r} = 0,$$

and $F_{N,r}(\pi_1(p)) = 0$ induces the relation

$$\sum_{(m_1, \dots, m_r) \in S_{N,r}} \left(\sum_{i=1}^{r-1} \delta \binom{m_2, \dots, m_i, m_{i+2}, \dots, m_r}{n_1, \dots, n_{i-1}, n_{i+2}, \dots, n_r} b_{n_i, n_{i+1}}^{m_1} \right) a_{m_1, \dots, m_r} = 0.$$

Subtracting the above two relations, one has

$$(3.29) \quad \sum_{(m_1, \dots, m_r) \in S_{N,r}} \left(-\delta \binom{m_1, \dots, m_r}{n_1, \dots, n_r} + \sum_{i=2}^{r-1} \delta \binom{m_2, \dots, m_i, m_{i+2}, \dots, m_r}{n_1, \dots, n_{i-1}, n_{i+2}, \dots, n_r} b_{n_i, n_{i+1}}^{m_1} \right) a_{m_1, \dots, m_r} = 0.$$

Furthermore, by definition of the space $\mathbf{W}_{N,r}$, every relation satisfied by a_{m_1, \dots, m_r} does not depend on the choices of $(m_3, \dots, m_r) \in S_{n, r-2}$ ($n = m_3 + \dots + m_r$), so that the relation (3.29) can be reduced to

$$\sum_{(m_1, m_2) \in S_{N-n, 2}} \left(\delta \binom{m_1, \dots, m_r}{n_1, \dots, n_r} - \sum_{i=2}^{r-1} \delta \binom{m_2, \dots, m_i, m_{i+2}, \dots, m_r}{n_1, \dots, n_{i-1}, n_{i+2}, \dots, n_r} b_{n_i, n_{i+1}}^{m_1} \right) a_{m_1, \dots, m_r} = 0.$$

Denote by $\alpha(n_1, \dots, n_r; m_3, \dots, m_r)$ the left-hand side of the above equation and $\text{bij}\{2, \dots, r\}$ the set of all bijections on the set $\{2, \dots, r\}$. Consider

$$\begin{aligned} \mathfrak{f}(m_3, \dots, m_r) &:= \sum_{\sigma \in \text{bij}\{2, \dots, r\}} \alpha(n_1, n_{\sigma(2)}, n_{\sigma(3)}, \dots, n_{\sigma(r)}; m_3, \dots, m_r) \\ &= \sum_{(m_1, m_2) \in S_{N-n, 2}} \sum_{\sigma \in \text{bij}\{2, \dots, r\}} \delta \binom{m_1, m_2, \dots, m_r}{n_1, n_{\sigma(2)}, \dots, n_{\sigma(r)}} a_{m_1, \dots, m_r} \\ &\quad - \sum_{(m_1, m_2) \in S_{N-n, 2}} \sum_{i=2}^{r-1} \left(\sum_{\sigma \in \text{bij}\{2, \dots, r\}} \delta \binom{m_2, m_3, \dots, m_i, m_{i+2}, \dots, m_r}{n_1, n_{\sigma(2)}, \dots, n_{\sigma(i-1)}, n_{\sigma(i+2)}, \dots, n_{\sigma(r)}} b_{n_{\sigma(i)}, n_{\sigma(i+1)}}^{m_1} \right) a_{m_1, \dots, m_r}, \end{aligned}$$

which is 0 for any $(m_3, \dots, m_r) \in S_{n, r-2}$. Notice that for each $\tau \in \text{bij}\{2, \dots, r\}$ there exists a unique $\tau' \in \text{bij}\{2, \dots, r\}$ such that $\tau(j) = \tau'(j)$ for $j \in \{2, \dots, i-1, i+2, \dots, r\}$, and $\tau(i) = \tau'(i+1)$, $\tau(i+1) = \tau'(i)$. This pairing, together with (3.4),

provides

$$\begin{aligned} & \delta\left(\begin{smallmatrix} m_2, m_3, \dots, m_i, m_{i+2}, \dots, m_r \\ n_1, n_{\sigma(2)}, \dots, n_{\tau(i-1)}, n_{\tau(i+2)}, \dots, n_{\tau(r)} \end{smallmatrix}\right) b_{n_{\tau(i)}, n_{\tau(i+1)}}^{m_1} \\ & + \delta\left(\begin{smallmatrix} m_2, m_3, \dots, m_i, m_{i+2}, \dots, m_r \\ n_1, n_{\tau'(2)}, \dots, n_{\tau'(i-1)}, n_{\tau'(i+2)}, \dots, n_{\tau'(r)} \end{smallmatrix}\right) b_{n_{\tau'(i)}, n_{\tau'(i+1)}}^{m_1} = 0. \end{aligned}$$

Then, for each $i \in \{2, \dots, r-1\}$, we have

$$\left(\sum_{\tau \in \text{bij}\{2, \dots, r\}} \delta\left(\begin{smallmatrix} m_2, m_3, \dots, m_i, m_{i+2}, \dots, m_r \\ n_1, n_{\tau(2)}, \dots, n_{\tau(i-1)}, n_{\tau(i+2)}, \dots, n_{\tau(r)} \end{smallmatrix}\right) b_{n_{\tau(i)}, n_{\tau(i+1)}}^{m_1} \right) a_{m_1, \dots, m_r} = 0,$$

and hence,

$$\mathfrak{f}(m_3, \dots, m_r) = \sum_{(m_1, m_2) \in S_{N-n, 2}} \sum_{\tau \in \text{bij}\{2, \dots, r\}} \delta\left(\begin{smallmatrix} m_1, m_2, \dots, m_r \\ n_1, n_{\tau(2)}, \dots, n_{\tau(r)} \end{smallmatrix}\right) a_{m_1, \dots, m_r}.$$

Letting $m_i = n_i$ for all $i \in \{3, \dots, r\}$, we obtain

$$\begin{aligned} 0 &= \mathfrak{f}(n_3, \dots, n_r) \\ &= \sum_{(m_1, m_2) \in S_{N-n, 2}} \sum_{\tau \in \text{bij}\{2, \dots, r\}} \delta\left(\begin{smallmatrix} m_1, m_2, n_3, \dots, n_r \\ n_1, n_{\tau(2)}, n_{\tau(3)}, \dots, n_{\tau(r)} \end{smallmatrix}\right) a_{m_1, m_2, n_3, \dots, n_r} \\ &= \left(\sum_{\tau \in \text{bij}\{2, \dots, r\}} \delta\left(\begin{smallmatrix} n_3, \dots, n_r \\ n_{\tau(3)}, \dots, n_{\tau(r)} \end{smallmatrix}\right) \right) a_{n_1, \dots, n_r}, \end{aligned}$$

which shows $a_{n_1, \dots, n_r} = 0$ for all $(n_1, \dots, n_r) \in S_{N, r}$. This completes the proof of Theorem 3.6. \square

As a corollary, the following inequalities are immediately obtained from (3.7).

Corollary 3.7. *For each integers $r \geq 3$, we have*

$$\sum_{N > 0} \dim_{\mathbb{Q}} \ker E_{N, r} x^N \geq \mathbb{S}(x) \cdot \mathbb{O}(x)^{r-2}.$$

Remark. Conjecturally, the dimension of $\ker E_{N, r}$ coincides with the coefficient of x^N in $\mathbb{S}(x) \cdot \mathbb{O}(x)^{r-2}$. Therefore, we may expect that $F_{N, r}$ gives a bijection from $\pi_1(\mathbf{W}_{N, r})$ to $\ker E_{N, r}$. We have a computational evidence up to $N = 35$ for this expectation, which was checked by direct calculations using Mathematica (within a few days).

4. PROOF OF THEOREM 1.4

This section is devoted to proving the inequality

$$(4.1) \quad \dim_{\mathbb{Q}} \ker C_{N,4} \geq \dim_{\mathbb{Q}} \ker E_{N,4} + \sum_{1 < n < N} \left(\dim_{\mathbb{Q}} \ker E_{N-n,3} + \text{rank } E_{n,2} \cdot \dim_{\mathbb{Q}} \ker E_{N-n,2} \right).$$

This inequality together with Corollaries 3.5 and 3.7 provides

$$\sum_{N>0} \dim_{\mathbb{Q}} \ker C_{N,4} x^N \geq 3\mathbb{S}(x) \cdot \mathbb{O}(x)^2 - \mathbb{S}(x)^2,$$

from which Theorem 1.4 follows. Then, Corollary 1.5 follows from Corollary 2.5.

4.1. Shuffle algebra. The formal graded vector space $\mathcal{Z}^{\text{odd}} := \bigoplus_{N,r \geq 0} \mathcal{Z}_{N,r}^{\text{odd}}$ has the structure of a commutative, bigraded \mathbb{Q} -algebra with respect to the series shuffle product developed by Hoffmann [8]: for example $\zeta_{\mathfrak{D}}(n_1)\zeta_{\mathfrak{D}}(n_2) = \zeta_{\mathfrak{D}}(n_1, n_2) + \zeta_{\mathfrak{D}}(n_2, n_1)$. To consider the formal one, let \mathbf{F} be the non-commutative polynomial algebra over \mathbb{Q} with one generator z_{2i+1} in every degree $2i+1$ for $i \geq 1$:

$$\mathbf{F} = \mathbb{Q}\langle z_3, z_5, z_7, \dots \rangle.$$

The bigraded piece of \mathbf{F} of weight N and depth r , which is the \mathbb{Q} -vector space spanned by the words $z_{n_1} \cdots z_{n_r}$ ($(n_1, \dots, n_r) \in S_{N,r}$), is denoted by $\mathbf{F}_{N,r}$. The empty word is regarded as $1 \in \mathbb{Q}$. The space \mathbf{F} has the structure of a commutative, bigraded algebra over \mathbb{Q} with respect to the shuffle product \mathfrak{III} :

$$(4.2) \quad z_{n_1} \cdots z_{n_r} \mathfrak{III} z_{n_{r+1}} \cdots z_{n_{r+s}} = \sum_{\substack{\sigma \in \mathfrak{S}_{r+s} \\ \sigma(1) < \cdots < \sigma(r) \\ \sigma(r+1) < \cdots < \sigma(r+s)}} z_{n_{\sigma^{-1}(1)}} \cdots z_{n_{\sigma^{-1}(r+s)}},$$

where \mathfrak{S}_n is the n -th symmetric group. We notice that the surjective linear map from \mathbf{F} to \mathcal{Z}^{odd} given by $z_{n_1} \cdots z_{n_r} \mapsto \zeta_{\mathfrak{D}}(n_1, \dots, n_r)$ becomes an algebra homomorphism.

4.2. Key identities. In this subsection, we show key identities, which are used to prove that the series shuffle product lifts every element of $\ker {}^t E_{N',r'}$ for $(N', r') < (N, 4)$ onto $\ker {}^t C_{N,4}$. The proof is done by tedious computations and postponed to the Appendix. We shall try to make an explanation of these identities using the notion of Brown's operator D_m (see Remark of the end of this subsection).

By definition (3.5), for $q \in \{2, \dots, r-1\}$ the matrix $E_{N,r}^{(q)}$ can be expressed as the direct sum of the form

$$\begin{aligned} E_{N,r}^{(q)} &= \bigoplus_{\substack{1 < p < N \\ (p_1, \dots, p_{r-q}) \in S_{p,r-q}}} E_{N-p,q} \\ &= \text{diag}(\underbrace{E_{3q,q}, \dots, E_{3q,q}}_{|S_{N-3q,r-q}|}, \underbrace{E_{3q+2,q}, \dots, E_{3q+2,q}}_{|S_{N-3q-2,r-q}|}, \dots, E_{N-3(r-q),q}). \end{aligned}$$

For $(p_1, \dots, p_{r-q}) \in S_{p,r-q}$ let us define the injective linear map $\Phi_{p_1, \dots, p_{r-q}}$ by

$$\begin{aligned} \Phi_{p_1, \dots, p_{r-q}} : \text{Vect}_{N-p,q} &\longrightarrow \text{Vect}_{N,r}, \\ (a_{n_1, \dots, n_q})_{(n_1, \dots, n_q) \in S_{N-p,q}} &\longmapsto \left(\delta_{\substack{p_1, \dots, p_{r-q} \\ n_1, \dots, n_{r-q}}} \cdot a_{n_{r-q+1}, \dots, n_r} \right)_{(n_1, \dots, n_r) \in S_{N,r}}. \end{aligned}$$

Then the above direct sum decomposition tells us

$$(4.3) \quad \ker {}^t E_{N,r}^{(q)} = \bigoplus_{\substack{1 < p < N \\ (p_1, \dots, p_{r-q}) \in S_{p,r-q}}} \Phi_{p_1, \dots, p_{r-q}}(\ker {}^t E_{N-p,q}).$$

Define an isomorphism $\pi_2 (= \pi_2^{(N,r)})$ of vector spaces Vect and \mathbf{F} by

$$\begin{aligned} \pi_2 : \text{Vect}_{N,r} &\longrightarrow \mathbf{F}_{N,r} \\ (a_{n_1, \dots, n_r})_{(n_1, \dots, n_r) \in S_{N,r}} &\longmapsto \sum_{(n_1, \dots, n_r) \in S_{N,r}} a_{n_1, \dots, n_r} z_{n_1} \cdots z_{n_r}. \end{aligned}$$

For every word $w \in \mathbf{F}_{N_1, r_1}$, we define the \mathbb{Q} -linear map Ψ_w on Vect , which corresponds to the shuffle product on \mathbf{F} , by the composition map

$$\begin{aligned} \Psi_w : \text{Vect}_{N_2, r_2} &\xrightarrow{\pi_2} \mathbf{F}_{N_2, r_2} \xrightarrow{\text{III } w} \mathbf{F}_{N_1+N_2, r_1+r_2} \xrightarrow{\pi_2^{-1}} \text{Vect}_{N_1+N_2, r_1+r_2} \\ v &\longmapsto \pi_2(v) \longmapsto \pi_2(v) \text{ III } w \longmapsto \pi_2^{-1}(\pi_2(v) \text{ III } w). \end{aligned}$$

For convenience, we write $\Psi_{n_1, \dots, n_r} = \Psi_{z_{n_1} \cdots z_{n_r}}$. We note that since the algebra \mathbf{F} is an integral domain, the map Ψ_{n_1, \dots, n_r} becomes an injection. Now we present our key identities whose proofs are postponed in the Appendix.

Lemma 4.1. (i) For each odd integer $p \geq 3$ and $v \in \ker {}^t E_{N-p,3}$, we have

$${}^t E_{N,4}(\Psi_p(v)) = \Phi_p(v) \in \ker {}^t E_{N,4}^{(3)}.$$

(ii) For each even integer $p \geq 6$, $(p_1, p_2) \in S_{p,2}$ and $v \in \ker {}^t E_{N-p,2}$, we have

$${}^t E_{N,4}^{(3)}({}^t E_{N,4}(\Psi_{p_1,p_2}(v))) = \sum_{(t_1,t_2) \in S_{p,2}} e_{(p_1,p_2)}^{(t_1,t_2)} \Phi_{t_1,t_2}(v) \in \ker {}^t E_{N,4}^{(2)}.$$

Corollary 4.2. (i) For each odd integer $p \geq 3$ and $v \in \ker {}^t E_{N-p,3}$, we have

$$\Psi_p(v) \in \ker {}^t C_{N,4}.$$

(ii) For each even integer $p \geq 6$, $(p_1, p_2) \in S_{p,2}$ and $v \in \ker {}^t E_{N-p,2}$, we have

$$\Psi_{p_1,p_2}(v) \in \ker {}^t C_{N,4}.$$

Proof. This follows from Lemma 4.1 and Proposition 3.3 (that is ${}^t C_{N,4} = {}^t E_{N,4} \cdot {}^t E_{N,4}^{(3)}$. ${}^t E_{N,4}^{(2)}$). \square

Remark. As above mentioned, the proof of Lemma 4.1 is boring. We suggest another proof using the context of Brown's derivation ∂_m . However, it still contains tedious calculations, which will be omitted.

Recall the graded version of Brown's operator $\text{gr}_r^{\mathfrak{D}} D_m$ on the depth-graded motivic MZVs (see §2.3). From definition, one can easily obtain

$$(4.4) \quad \text{gr}_r^{\mathfrak{D}} D_m(\zeta_{\mathfrak{D}}^{\mathfrak{m}}(n_1, \dots, n_r)) = \sum_{(m_1, \dots, m_r) \in S_{N,r}} \delta_{(m)}^{(m_1)} e_{(n_1, \dots, n_r)}^{(m_1, \dots, m_r)} \zeta_{m_1} \otimes \zeta_{\mathfrak{D}}^{\mathfrak{m}}(m_2, \dots, m_r).$$

We now define operators d_m for odd $m > 1$, which are a formal version of $\text{gr}_r^{\mathfrak{D}} D_m$. For $z_{n_1} \cdots z_{n_r} \in \mathbf{F}_{N,r}$, let

$$d_m(z_{n_1} \cdots z_{n_r}) = \sum_{(m_1, \dots, m_r) \in S_{N,r}} \delta_{(m)}^{(m_1)} e_{(n_1, \dots, n_r)}^{(m_1, \dots, m_r)} z_{m_1} \otimes z_{m_2} \cdots z_{m_r}$$

and $d_m(z_n) = \delta_{(n)}^{(m)} z_n \otimes 1$. Then we obtain the \mathbb{Q} -linear map $d_m : \mathbf{F}_{N,r} \rightarrow \mathbb{Q} z_m \otimes \mathbf{F}_{N-m,r-1}$ by linearity. Set $d_{<N} := \sum_{1 < m < N: \text{odd}} d_m$:

$$d_{<N} : \mathbf{F}_{N,r} \longrightarrow \mathbf{F}_{N,r}^{(r-1)},$$

where we write $\mathbf{F}_{N,r}^{(r-1)} = \bigoplus_{1 < m < N} \mathbf{F}_{m,1} \otimes \mathbf{F}_{N-m,r-1}$. In general, we put for $2 \leq q \leq r$

$$\mathbf{F}_{N,r}^{(q)} = \bigoplus_{\substack{1 < p < N \\ (p_1, \dots, p_{r-q}) \in S_{p,r-q}}} \underbrace{\mathbf{F}_{p_1,1} \otimes \cdots \otimes \mathbf{F}_{p_{r-q},1}}_{r-q} \otimes \mathbf{F}_{N-p,q}.$$

(Notice $\mathbf{F}_{N,r}^{(r)} = \mathbf{F}_{N,r}$.) Define isomorphisms $\pi_2^{(q)} (= \pi_2^{(N,r,q)})$ for $2 \leq q \leq r$ by

$$\pi_2^{(q)} : \mathbf{Vect}_{N,r} \longrightarrow \mathbf{F}_{N,r}^{(q)}$$

$$(a_{n_1, \dots, n_r})_{(n_1, \dots, n_r) \in S_{N,r}} \longmapsto \sum_{(n_1, \dots, n_r) \in S_{N,r}} a_{n_1, \dots, n_r} z_{n_1} \otimes \cdots \otimes z_{n_{r-q}} \otimes z_{n_{r-q+1}} \cdots z_{n_r}.$$

(Notice $\pi_2^{(r)} = \pi_2$.) Set $d_{<N}^{(q)} := \text{id}^{\otimes(r-q)} \otimes d_{<N}$ for $2 \leq q \leq r$, so then $d_{<N}^{(r)} = d_{<N}$. By definition, for $3 \leq q \leq r$ one immediately finds that the following diagram commutes:

$$\begin{array}{ccc} \text{Vect}_{N,r} & \xrightarrow{\pi_2^{(q)}} & \mathbf{F}_{N,r}^{(q)} \\ \downarrow {}^t E_{N,r}^{(q)} & & \downarrow d_{<N}^{(q)} \\ \text{Vect}_{N,r} & \xrightarrow{\pi_2^{(q-1)}} & \mathbf{F}_{N,r}^{(q-1)} \end{array}$$

We notice that the linear map $C_{N,r}$ can be written as the composition map $(\pi_2^{(2)})^{-1} \circ d_{<N}^{(2)} \circ \cdots \circ d_{<N}^{(r-1)} \circ d_{<N}^{(r)} \circ \pi_2^{(r)}$:

$$\pi_2^{(2)}(C_{N,r}(v)) = d_{<N}^{(2)} \circ \cdots \circ d_{<N}^{(r-1)} \circ d_{<N}^{(r)}(\pi_2^{(r)}(v)).$$

We are expecting that the map d_m is a derivation, i.e. $d_m(w_1 \boxplus w_2) = d_m(w_1) \boxplus (1 \otimes w_2) + d_m(w_2) \boxplus (1 \otimes w_1)$, however we were not able to prove this (whereas we can prove that d_m is a derivation for the shuffle product of iterated integrals, since the coefficient $e_{n_1, \dots, n_r}^{(m_1, \dots, m_r)}$ in the definition is obtained from the shuffle product). One can at least check the following (so the proof is also by boring computations, we omit it):

$$\begin{aligned} d_m(z_p \boxplus z_{n_1} z_{n_2} z_{n_3}) &= d_m(z_p) \boxplus (1 \otimes z_{n_1} z_{n_2} z_{n_3}) + d_m(z_{n_1} z_{n_2} z_{n_3}) \boxplus (1 \otimes z_p), \\ d_m(z_{p_1} z_{p_2} \boxplus z_{n_1} z_{n_2}) &= d_m(z_{p_1} z_{p_2}) \boxplus (1 \otimes z_{n_1} z_{n_2}) + d_m(z_{n_1} z_{n_2}) \boxplus (1 \otimes z_{p_1} z_{p_2}). \end{aligned}$$

Using the first identity, one can compute

$$\begin{aligned} {}^t E_{N,4}(\Psi_p(v)) &= (\pi_2^{(3)})^{-1} \circ d_{<N} \circ \pi_2(\Psi_p(v)) = (\pi_2^{(3)})^{-1} \circ d_{<N}(z_p \boxplus \pi_2(v)) \\ &= (\pi_2^{(3)})^{-1} (d_{<N}(z_p) \boxplus (1 \otimes \pi_2(v)) + d_{<N}(\pi_2(v)) \boxplus (1 \otimes z_p)) \\ &= (\pi_2^{(3)})^{-1} (z_p \otimes \pi_2(v)) = \Phi_p(v), \end{aligned}$$

where for the last equality we have used $d_{<N}(\pi_2(v)) = 0$ which is equivalent to the assumption ${}^t E_{N-p,3}(v) = 0$. This gives a proof of Lemma 4.1 (i). The same proof works for the formula (ii) of Lemma 4.1, so we omit this.

We remark that if it is shown that d_m becomes a derivation, one can prove for $v \in \ker {}^t E_{N-p,r-q}$ and $(p_1, \dots, p_q) \in S_{p,q}$

$${}^t(E_{N,r} E_{N,r}^{(r-1)} \dots E_{N,r}^{(r-q+1)})(\Psi_{p_1, \dots, p_q}(v)) = \sum_{(t_1, \dots, t_q) \in S_{p,q}} c_{(p_1, \dots, p_q)}^{(t_1, \dots, t_q)} \Phi_{t_1, \dots, t_q}(v),$$

which implies $\Psi_{p_1, \dots, p_q}(v) \in \ker {}^t C_{N,r}$.

4.3. Proof of Theorem 1.4. From Corollary 4.2, one has

$$\ker {}^t C_{N,4} \supset \ker {}^t E_{N,4} + \sum_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3}) + \sum_{\substack{1 < p < N \\ (p_1, p_2) \in S_{p,2}}} \Psi_{p_1, p_2}(\ker {}^t E_{N-p,2}).$$

For $1 < p < N$ we now consider the \mathbb{Q} -vector space $A_N^{(p)} \subset \mathbf{Vect}_{N,4}$ spanned by all elements of the form $\sum_{(p_1, p_2) \in S_{p,2}} a_{p_1, p_2} \Psi_{p_1, p_2}(v)$, where $(a_{p_1, p_2})_{(p_1, p_2) \in S_{p,2}} \in \text{Im } {}^t E_{p,2}$ and $v \in \ker {}^t E_{N-p,2}$:

$$A_N^{(p)} = \left\langle \sum_{(p_1, p_2) \in S_{p,2}} a_{p_1, p_2} \Psi_{p_1, p_2}(v) \mid (a_{p_1, p_2})_{(p_1, p_2) \in S_{p,2}} \in \text{Im } {}^t E_{p,2}, v \in \ker {}^t E_{N-p,2} \right\rangle_{\mathbb{Q}}.$$

Our goal is to show the following:

$$(4.5) \quad \ker {}^t C_{N,4} \supset \ker {}^t E_{N,4} \oplus \bigoplus_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3}) \oplus \bigoplus_{1 < p < N} A_N^{(p)}.$$

Before proving (4.5), we show (4.1) assuming (4.5). From the injectivity of the map Ψ_p , one has

$$(4.6) \quad \dim_{\mathbb{Q}} \Psi_p(\ker {}^t E_{N-p,3}) = \dim_{\mathbb{Q}} \ker {}^t E_{N-p,3}.$$

By using the fact that the algebra \mathbf{F} is isomorphic to a certain polynomial algebra, one can prove the following lemma (the proof will be postponed to the Appendix).

Lemma 4.3. *Let \mathbf{B}_1 and \mathbf{B}_2 be subspaces of $\mathbf{F}_{N_1,2}$ and $\mathbf{F}_{N_2,2}$ respectively. Assume that $\mathbf{B}_1 \cap \mathbf{B}_2 = \{0\}$. Then the dimension of the space spanned by all elements composing of two products of $w_1 \in \mathbf{B}_1$ and $w_2 \in \mathbf{B}_2$ is equal to $\dim_{\mathbb{Q}} \mathbf{B}_1 \cdot \dim_{\mathbb{Q}} \mathbf{B}_2$:*

$$\dim_{\mathbb{Q}} \langle w_1 \boxplus w_2 \mid w_1 \in \mathbf{B}_1, w_2 \in \mathbf{B}_2 \rangle_{\mathbb{Q}} = \dim_{\mathbb{Q}} \mathbf{B}_1 \cdot \dim_{\mathbb{Q}} \mathbf{B}_2.$$

From this lemma, one immediately obtains

$$(4.7) \quad \dim_{\mathbb{Q}} \pi_2(A_N^{(p)}) = \text{rank } {}^t E_{p,2} \cdot \dim_{\mathbb{Q}} \ker {}^t E_{N-p,2}.$$

Then, the inequality (4.1) follows from (4.6), (4.7) and (4.5). We now prove (4.5).

Proof of (4.5). For any element $v = \sum_{1 < p < N} \Psi_p(v_p) \in \sum_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3})$, using Lemma 4.1 (i), one has

$$\begin{aligned} {}^t E_{N,4}(v) &= \sum_{1 < p < N} {}^t E_{N,4}(\Psi_p(v_p)) = \sum_{1 < p < N} \Phi_p(v_p) \in \ker {}^t E_{N,4}^{(3)} \\ &\left(= \bigoplus_{1 < p < N} \Phi_p(\ker {}^t E_{N-p,3}), \text{ by (4.3)} \right). \end{aligned}$$

Therefore, the injectivity of the map Φ_p shows that ${}^t E_{N,4}(v) = 0$ if and only if $v_p = 0$ for all $1 < p < N$. So then $v = 0 \Rightarrow v_p = 0$ for all $1 < p < N$ implies

$$\sum_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3}) = \bigoplus_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3}),$$

and ${}^t E_{N,4}(v) = 0 \Rightarrow v = 0$ shows

$$\ker {}^t E_{N,4} \oplus \bigoplus_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3}).$$

We fix a basis of $\ker {}^t E_{N-p,2}$ denoted by $\{w_i^{(p)}\}_{i=1}^{k_p}$. Then for any $v_p \in \mathbf{A}_N^{(p)}$ one can express

$$v_p = \sum_{i=1}^{k_p} \sum_{(p_1, p_2) \in S_{p,2}} a_{p_1, p_2}^{(i)} \Psi_{p_1, p_2}(w_i^{(p)})$$

with some elements $(a_{p_1, p_2}^{(i)})_{(p_1, p_2) \in S_{p,2}} \in \text{Im } {}^t E_{p,2}$ ($1 \leq i \leq k_p$). For any

$$v = \sum_{1 < p < N} \alpha_p v_p = \sum_{1 < p < N} \alpha_p \sum_{i=1}^{k_p} \sum_{(p_1, p_2) \in S_{p,2}} a_{p_1, p_2}^{(i)} \Psi_{p_1, p_2}(w_i^{(p)}) \in \sum_{1 < p < N} \mathbf{A}_N^{(p)},$$

using Lemma 4.1 (ii), one has

$$\begin{aligned} {}^t E_{N,4}^{(3)}({}^t E_{N,4}(v)) &= \sum_{1 < p < N} \sum_{i=1}^{k_p} \sum_{(t_1, t_2) \in S_{p,2}} \alpha_p \left(\sum_{(p_1, p_2) \in S_{p,2}} a_{p_1, p_2}^{(i)} e_{(p_1, p_2)}^{(t_1, t_2)} \right) \Phi_{t_1, t_2}(w_i^{(p)}) \\ &\in \ker {}^t E_{N,4}^{(2)} \left(= \bigoplus_{\substack{1 < p < N \\ (t_1, t_2) \in S_{p,2}}} \Phi_{t_1, t_2}(\ker {}^t E_{N-p,2}), \text{ by (4.3)} \right). \end{aligned}$$

For each $1 < p < N$ the set $\{\Phi_{t_1, t_2}(w_i^{(p)}) \mid 1 \leq i \leq k_p, (t_1, t_2) \in S_{p,2}\}$ is linearly independent over \mathbb{Q} . This shows that ${}^t E_{N,4}^{(3)}({}^t E_{N,4}(v)) = 0$ if and only if for all $1 < p < N$, each coefficient of $\Phi_{t_1, t_2}(w_i^{(p)})$ for $(t_1, t_2) \in S_{p,2}$ and $1 \leq i \leq k_p$, which is $\alpha_p \sum_{(p_1, p_2) \in S_{p,2}} a_{p_1, p_2}^{(i)} e_{(p_1, p_2)}^{(t_1, t_2)}$, is 0. Since if $\sum_{(p_1, p_2) \in S_{p,2}} a_{p_1, p_2}^{(i)} e_{(p_1, p_2)}^{(t_1, t_2)} = 0$ for all $(t_1, t_2) \in$

$S_{p,2}$, then $0 = (a_{p_1,p_2}^{(i)})_{(p_1,p_2) \in S_{p,2}}$ because $(a_{p_1,p_2}^{(i)})_{(p_1,p_2) \in S_{p,2}} \in \ker {}^t E_{p,2} \cap \text{Im } {}^t E_{p,2} = \{0\}$, it turns out to be $v_p = 0$ whenever $\alpha_p \neq 0$. Therefore, assuming $0 = v = \sum w_p \in \sum A_N^{(p)}$, we have $w_p = 0$ for each $1 < p < N$. This gives

$$\sum_{1 < p < N} A_N^{(p)} = \bigoplus_{1 < p < N} A_N^{(p)}.$$

We also have ${}^t E_{N,4}^{(3)}({}^t E_{N,4}(v)) = 0 \Rightarrow v = 0$, which is used below. The reminder is to show that

$$\left(\ker {}^t E_{N,4} \oplus \bigoplus_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3}) \right) \cap \bigoplus_{1 < p < N} A_N^{(p)} = \{0\}.$$

It follows from the above discussion that the map ${}^t E_{N,4} : \bigoplus_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3}) \rightarrow \ker {}^t E_{N,4}^{(3)}$ is an injection. Since from the injectivity of the map Ψ_p and (4.3) we have

$$\dim_{\mathbb{Q}} \left(\bigoplus_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3}) \right) = \sum_{1 < p < N} \dim_{\mathbb{Q}} \ker {}^t E_{N-p,3} = \dim_{\mathbb{Q}} \ker {}^t E_{N,4}^{(3)},$$

one finds that the above map becomes an isomorphism. Thus for $v \in \ker {}^t(E_{N,4}^{(3)} \cdot E_{N,4})$ we have

$${}^t E_{N,4}(v) \in \ker {}^t E_{N,4}^{(3)} = {}^t E_{N,4} \left(\bigoplus_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3}) \right),$$

which implies that there exists $v' \in \bigoplus_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3})$ such that $v - v' \in \ker {}^t E_{N,4}$, so that $v \in \ker {}^t E_{N,4} \oplus \bigoplus_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3})$. Thereby $\ker {}^t(E_{N,4}^{(3)} \cdot E_{N,4}) = \ker {}^t E_{N,4} \oplus \bigoplus_{1 < p < N} \Psi_p(\ker {}^t E_{N-p,3})$. We have already shown that if $v \in \bigoplus_{1 < p < N} A_N^{(p)}$ satisfies ${}^t E_{N,4}^{(3)}({}^t E_{N,4}(v)) = 0$, then $v = 0$. This completes the proof. \square

Remark. In general, we conjecture that the dimension of the kernel of the matrix $C_{N,r}$ is given by

$$\dim \ker {}^t C_{N,r} \stackrel{?}{=} \dim \ker {}^t E_{N,r} + \sum_{1 \leq q \leq r-2} \left(\sum_{1 < p < N} \text{rank } C_{p,q} \cdot \dim \ker {}^t E_{N-p,r-q} \right).$$

Then, by induction on r , the uneven part of motivic Broadhurst-Kreimer conjecture (2.4) follows from the (conjectural) equality $\sum_{k \geq 0} \dim \ker E_{N,r} x^k = \mathbb{O}(x)^{r-2} \mathbb{S}(x)$ (or equivalently, the surjectivity of the map $F_{N,r}$ in Theorem 3.6).

APPENDIX

In this appendix, we prove Lemmas 4.1 and 4.3.

Proof of Lemma 4.1. Lemma 4.1 is shown by direct calculations. We only prove (ii). Denote $v = (a_{n_1, n_2})_{(n_1, n_2) \in S_{N-p, 2}}$. Then from (4.2) each components of the row vector $\Psi_{p_1, p_2}(v) = (A_{n_1, \dots, n_4}^{(p_1, p_2)})_{(n_1, \dots, n_4) \in S_{N, 4}}$ can be written in the form

$$\begin{aligned} A_{n_1, \dots, n_4}^{(p_1, p_2)} &= \delta_{p_1, p_2}^{(n_1, n_2)} a_{n_3, n_4} + \delta_{p_1, p_2}^{(n_1, n_3)} a_{n_2, n_4} + \delta_{p_1, p_2}^{(n_1, n_4)} a_{n_2, n_3} \\ &\quad + \delta_{p_1, p_2}^{(n_2, n_3)} a_{n_1, n_4} + \delta_{p_1, p_2}^{(n_2, n_4)} a_{n_1, n_3} + \delta_{p_1, p_2}^{(n_3, n_4)} a_{n_1, n_2}. \end{aligned}$$

The (n_1, \dots, n_4) -th entry of the vector ${}^t E_{N, 4}(\Psi_{p_1, p_2}(v))$ can be computed as follows.

$$\begin{aligned} &\sum_{(m_1, \dots, m_4) \in S_{N, 4}} A_{m_1, \dots, m_4}^{(p_1, p_2)} (\delta_{m_1, \dots, m_4}^{(n_1, \dots, n_4)} + \delta_{m_3, m_4}^{(n_3, n_4)}) b_{m_1, m_2}^{n_1} \\ &\quad + \delta_{m_1, m_4}^{(n_2, n_4)} b_{m_2, m_3}^{n_1} + \delta_{m_1, m_2}^{(n_2, n_3)} b_{m_3, m_4}^{n_1}) \\ &= \delta_{p_1, p_2}^{(n_1, n_2)} a_{n_3, n_4} + \delta_{p_1, p_2}^{(n_1, n_3)} a_{n_2, n_4} + \delta_{p_1, p_2}^{(n_1, n_4)} a_{n_2, n_3} \\ &\quad + \delta_{p_1, p_2}^{(n_2, n_3)} a_{n_1, n_4} + \delta_{p_1, p_2}^{(n_2, n_4)} a_{n_1, n_3} + \delta_{p_1, p_2}^{(n_3, n_4)} a_{n_1, n_2} \end{aligned} \quad (4.8)$$

$$\begin{aligned} &+ \sum_{(m_1, \dots, m_4) \in S_{N, 4}} \left(\delta_{p_1, p_2}^{(m_1, m_2)} \delta_{m_3, m_4}^{(n_3, n_4)} b_{m_1, m_2}^{n_1} a_{m_3, m_4} + \delta_{p_1, p_2}^{(m_1, m_3)} \delta_{m_3, m_4}^{(n_3, n_4)} b_{m_1, m_2}^{n_1} a_{m_2, m_4} \right. \\ &\quad \left. + \delta_{p_1, p_2}^{(m_1, m_4)} \delta_{m_3, m_4}^{(n_3, n_4)} b_{m_1, m_2}^{n_1} a_{m_2, m_3} + \delta_{p_1, p_2}^{(m_2, m_3)} \delta_{m_3, m_4}^{(n_3, n_4)} b_{m_1, m_2}^{n_1} a_{m_1, m_4} \right. \end{aligned} \quad (4.9)$$

$$\begin{aligned} &\quad \left. + \delta_{p_1, p_2}^{(m_2, m_4)} \delta_{m_3, m_4}^{(n_3, n_4)} b_{m_1, m_2}^{n_1} a_{m_1, m_3} + \delta_{p_1, p_2}^{(m_3, m_4)} \delta_{m_3, m_4}^{(n_3, n_4)} b_{m_1, m_2}^{n_1} a_{m_1, m_2} \right) \end{aligned} \quad (4.10)$$

$$\begin{aligned} &\quad + \delta_{p_1, p_2}^{(m_1, m_2)} \delta_{m_1, m_4}^{(n_2, n_4)} b_{m_2, m_3}^{n_1} a_{m_3, m_4} + \delta_{p_1, p_2}^{(m_1, m_3)} \delta_{m_1, m_4}^{(n_2, n_4)} b_{m_2, m_3}^{n_1} a_{m_2, m_4} \\ &\quad + \delta_{p_1, p_2}^{(m_1, m_4)} \delta_{m_1, m_4}^{(n_2, n_4)} b_{m_2, m_3}^{n_1} a_{m_2, m_3} + \delta_{p_1, p_2}^{(m_2, m_3)} \delta_{m_1, m_4}^{(n_2, n_4)} b_{m_2, m_3}^{n_1} a_{m_1, m_4} \end{aligned} \quad (4.11)$$

$$\begin{aligned} &\quad + \delta_{p_1, p_2}^{(m_2, m_4)} \delta_{m_1, m_4}^{(n_2, n_4)} b_{m_2, m_3}^{n_1} a_{m_1, m_3} + \delta_{p_1, p_2}^{(m_3, m_4)} \delta_{m_1, m_4}^{(n_2, n_4)} b_{m_2, m_3}^{n_1} a_{m_1, m_2} \end{aligned} \quad (4.12)$$

$$\begin{aligned} &\quad + \delta_{p_1, p_2}^{(m_1, m_2)} \delta_{m_1, m_2}^{(n_2, n_3)} b_{m_3, m_4}^{n_1} a_{m_3, m_4} + \delta_{p_1, p_2}^{(m_1, m_3)} \delta_{m_1, m_2}^{(n_2, n_3)} b_{m_3, m_4}^{n_1} a_{m_2, m_4} \\ &\quad + \delta_{p_1, p_2}^{(m_1, m_4)} \delta_{m_1, m_2}^{(n_2, n_3)} b_{m_3, m_4}^{n_1} a_{m_2, m_3} + \delta_{p_1, p_2}^{(m_2, m_3)} \delta_{m_1, m_2}^{(n_2, n_3)} b_{m_3, m_4}^{n_1} a_{m_1, m_4} \end{aligned} \quad (4.13)$$

$$\begin{aligned} &\quad + \delta_{p_1, p_2}^{(m_2, m_4)} \delta_{m_1, m_2}^{(n_2, n_3)} b_{m_3, m_4}^{n_1} a_{m_1, m_3} + \delta_{p_1, p_2}^{(m_3, m_4)} \delta_{m_1, m_2}^{(n_2, n_3)} b_{m_3, m_4}^{n_1} a_{m_1, m_2} \Big). \end{aligned} \quad (4.14)$$

$$\begin{aligned} &\quad + \delta_{p_1, p_2}^{(m_2, m_4)} \delta_{m_1, m_2}^{(n_2, n_3)} b_{m_3, m_4}^{n_1} a_{m_1, m_3} + \delta_{p_1, p_2}^{(m_3, m_4)} \delta_{m_1, m_2}^{(n_2, n_3)} b_{m_3, m_4}^{n_1} a_{m_1, m_2} \Big). \end{aligned} \quad (4.15)$$

One can find easily that the sum of second term of (4.8) plus the sum of second term of (4.9) is 0 because of (3.4). In the same way, we can derive (4.9)(1) + (4.10)(1) = (4.11)(1) + (4.11)(2) = (4.12)(1) + (4.12)(2) = (4.13)(2) + (4.14)(1) = (4.14)(2) + (4.15)(1) = 0, where the notation, for example, (4.9)(1) means the first term of (4.9). Then the above equation can be reduced to

$$\begin{aligned} & \delta\binom{n_1, n_2}{p_1, p_2} a_{n_3, n_4} + \delta\binom{n_1, n_3}{p_1, p_2} a_{n_2, n_4} + \delta\binom{n_1, n_4}{p_1, p_2} a_{n_2, n_3} + \delta\binom{n_2, n_3}{p_1, p_2} a_{n_1, n_4} + \delta\binom{n_2, n_4}{p_1, p_2} a_{n_1, n_3} + \delta\binom{n_3, n_4}{p_1, p_2} a_{n_1, n_2} \\ & + b_{p_1, p_2}^{n_1} (a_{n_3, n_4} + a_{n_2, n_4} + a_{n_2, n_3}) + \sum_{(m_1, \dots, m_4) \in S_{N,4}} \left(\delta\binom{m_3, m_4}{p_1, p_2} \delta\binom{n_3, n_4}{m_3, m_4} b_{m_1, m_2}^{n_1} a_{m_1, m_2} \right. \\ & \left. + \delta\binom{m_1, m_4}{p_1, p_2} \delta\binom{n_2, n_4}{m_1, m_4} b_{m_2, m_3}^{n_1} a_{m_2, m_3} + \delta\binom{m_1, m_2}{p_1, p_2} \delta\binom{n_2, n_3}{m_1, m_2} b_{m_3, m_4}^{n_1} a_{m_3, m_4} \right). \end{aligned}$$

(Note $a_{n_1, n_2} = 0$ whenever $n_1 + n_2 \neq N - p$.) From the assumption ${}^t E_{N-p, 2}(v) = 0$, one has for $(n_1, \dots, n_4) \in S_{N,4}$

$$\sum_{(m_1, \dots, m_4) \in S_{N,4}} \delta\binom{m_3, m_4}{p_1, p_2} \delta\binom{n_3, n_4}{m_3, m_4} (\delta\binom{n_1, n_2}{m_1, m_2} + b_{m_1, m_2}^{n_1}) a_{m_1, m_2} = 0,$$

which entails

$$\begin{aligned} & {}^t E_{N,4}(\Psi_{p_1, p_2}(v)) = \\ & \left(b_{p_1, p_2}^{n_1} (a_{n_3, n_4} + a_{n_2, n_4} + a_{n_2, n_3}) + \delta\binom{n_1, n_2}{p_1, p_2} a_{n_3, n_4} + \delta\binom{n_1, n_3}{p_1, p_2} a_{n_2, n_4} + \delta\binom{n_1, n_4}{p_1, p_2} a_{n_2, n_3} \right)_{(n_1, \dots, n_4) \in S_{N,4}}. \end{aligned}$$

We denote by $B_{n_1, \dots, n_4}^{(p_1, p_2)}$ the (n_1, \dots, n_4) -th entry of the above. Then the (n_1, \dots, n_4) -th entry of the vector ${}^t E_{N,4}^{(3)}({}^t E_{N,4}(\Psi_{p_1, p_2}(v)))$ can be computed as follows:

$$\begin{aligned} & \sum_{(m_1, \dots, m_4) \in S_{N,4}} B_{m_1, \dots, m_4}^{(p_1, p_2)} (\delta\binom{n_1, \dots, n_4}{m_1, \dots, m_4} + \delta\binom{n_1, n_4}{m_1, m_4} b_{m_2, m_3}^{n_2} + \delta\binom{n_1, n_3}{m_1, m_2} b_{m_3, m_4}^{n_2}) \\ (4.16) \quad & = b_{p_1, p_2}^{n_1} (a_{n_3, n_4} + a_{n_2, n_4} + a_{n_2, n_3}) + \delta\binom{n_1, n_2}{p_1, p_2} a_{n_3, n_4} + \delta\binom{n_1, n_3}{p_1, p_2} a_{n_2, n_4} + \delta\binom{n_1, n_4}{p_1, p_2} a_{n_2, n_3} \\ (4.17) \quad & + \sum_{(m_1, \dots, m_4) \in S_{N,4}} \delta\binom{n_1, n_4}{m_1, m_4} b_{m_2, m_3}^{n_2} (b_{p_1, p_2}^{m_1} (a_{m_3, m_4} + a_{m_2, m_4} + a_{m_2, m_3}) \\ (4.18) \quad & + \delta\binom{m_1, m_2}{p_1, p_2} a_{m_3, m_4} + \delta\binom{m_1, m_3}{p_1, p_2} a_{m_2, m_4} + \delta\binom{m_1, m_4}{p_1, p_2} a_{m_2, m_3}) \end{aligned}$$

$$(4.19) \quad + \sum_{(m_1, \dots, m_4) \in S_{N,4}} \delta \binom{n_1, n_3}{m_1, m_2} b_{m_3, m_4}^{n_2} (b_{p_1, p_2}^{m_1} (a_{m_3, m_4} + a_{m_2, m_4} + a_{m_2, m_3}))$$

$$(4.20) \quad + \delta \binom{m_1, m_2}{p_1, p_2} a_{m_3, m_4} + \delta \binom{m_1, m_3}{p_1, p_2} a_{m_2, m_4} + \delta \binom{m_1, m_4}{p_1, p_2} a_{m_2, m_3}.$$

We note that from ${}^t E_{N-p,2}(v) = 0$ one finds

$$(4.16)(2) + (4.19)(1) = \sum_{(m_1, \dots, m_4) \in S_{N,4}} \delta \binom{n_1, n_3}{m_1, m_2} b_{p_1, p_2}^{m_1} \left(\delta \binom{n_2, n_4}{m_3, m_4} + b_{m_3, m_4}^{n_2} \right) a_{m_3, m_4} = 0,$$

where we used the same notation as before. In the same way, we also have $(4.16)(3) + (4.17)(3) = 0$. It follows from ${}^t E_{N-p,2}(v) = 0$ that $(4.16)(5) + (4.20)(1) = (4.16)(6) + (4.18)(3) = 0$. Using $b_{n,n'}^m + b_{n',n}^m = 0$, it follows $(4.18)(1) + (4.18)(2) = (4.20)(2) + (4.20)(3) = 0$, and also changing variables $m_3 \leftrightarrow m_4$ shows $(4.19)(2) + (4.19)(3) = 0$, and changing variables $m_2 \leftrightarrow m_3$ gives $(4.17)(1) + (4.17)(2) = 0$. So, the remainder is

$$\left(\delta \binom{n_1, n_2}{p_1, p_2} + b_{p_1, p_2}^{n_1} \right) a_{n_3, n_4} = e \binom{n_1, n_2}{p_1, p_2} a_{n_3, n_4} = \sum_{(t_1, t_2) \in S_{p,2}} e \binom{t_1, t_2}{p_1, p_2} \delta \binom{t_1, t_2}{n_1, n_2} a_{n_3, n_4},$$

which shows

$${}^t E_{N,4}^{(3)}({}^t E_{N,4}(\Psi_{p_1, p_2}(v))) = \sum_{(t_1, t_2) \in S_{p,2}} e \binom{t_1, t_2}{p_1, p_2} \Phi_{t_1, t_2}(v).$$

From (4.3) one has ${}^t E_{N,4}^{(3)}({}^t E_{N,4}(\Psi_{p_1, p_2}(v))) \in \ker {}^t E_{N,4}^{(2)}$. Thus we complete the proof of (ii). \square

For the proof of Lemma 4.3, the important fact is that the algebra \mathbf{F} is isomorphic to the polynomial algebra in the Lyndon words (see [12, Theorem in p.589]). Define a total ordering for the set $\{z_3, z_5, \dots\}$ as $z_{n_1} < z_{n_2}$ for $n_1 < n_2$. The Lyndon word w is defined as an element of $\{z_3, z_5, \dots\}^\times$ such that w is smaller than every strict right factor in the lexicographic ordering $w < v$ if $w = uv$, where $u, v \neq \emptyset$. Hereafter, the basis of $\mathbf{F}_{N,r}$ consisting of monomials in the Lyndon words is called the Lyndon basis. For example, the Lyndon basis of the \mathbb{Q} -vector space $\mathbf{F}_{N,2}$ is given by the set $\{z_{n_1} z_{n_2} \mid (n_1, n_2) \in L_N\} \cup \{z_{n_1} \amalg z_{n_2} \mid (n_1, n_2) \in L_N^*\}$, where we set

$$L_N := \{(n_1, n_2) \in S_{N,2} \mid n_1 < n_2\} \text{ and } L_N^* := \{(n_1, n_2) \in S_{N,2} \mid n_1 \leq n_2\}.$$

To prove Lemma 4.3, we have only to show the following lemma (the rest follows from the standard linear algebra).

Lemma . *For integers $N_1, N_2 > 0$, the set $\{\alpha_i \sqcup \beta_j \mid 1 \leq i \leq g, 1 \leq j \leq h\}$ is linearly independent over \mathbb{Q} , where we denote by $\{\alpha_1, \dots, \alpha_g\}$ (resp. $\{\beta_1, \dots, \beta_h\}$) the Lyndon basis of $\mathbf{F}_{N_1,2}$ (resp. $\mathbf{F}_{N_2,2}$). The set $\{\alpha_i \sqcup \alpha_j \mid 1 \leq i \leq j \leq g\}$ is also linearly independent.*

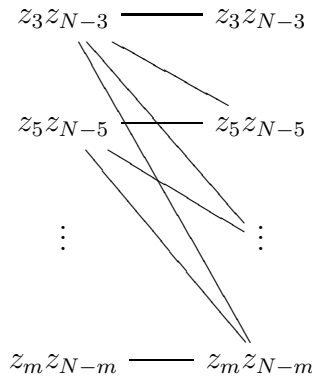
Proof. Notice that each of the sets

$$\begin{aligned} \Pi_N &:= \{z_{n_1} \sqcup z_{n_2} \sqcup z_{n_3} \sqcup z_{n_4} \mid (n_1, \dots, n_4) \in S_{N,4}, n_1 \leq n_2 \leq n_3 \leq n_4\}, \\ \Pi_{N_1, N_2}^{(1)} &:= \{z_{n_1} \sqcup z_{n_2} \sqcup z_{n_3} z_{n_4} \mid (n_1, n_2) \in L_{N_1}^*, (n_3, n_4) \in L_{N_2}\}, \\ \Pi_{N_1, N_2}^{(2)} &:= \begin{cases} \{z_{n_1} z_{n_2} \sqcup z_{n_3} z_{n_4} \mid (n_1, n_2) \in L_{N_1}, (n_3, n_4) \in L_{N_2}\} & \text{if } N_1 \neq N_2 \\ \{z_{n_1} z_{n_2} \sqcup z_{n_3} z_{n_4} \mid (n_1, n_2) \in L_{N_1}, (n_3, n_4) \in L_{N_2}, n_1 \leq n_3\} & \text{if } N_1 = N_2 \end{cases} \end{aligned}$$

is a subset of the Lyndon basis of $\mathbf{F}_{N,4}$. Therefore, the set

$$\Pi_N \cup \bigcup_{(N_1, N_2) \in S_{N,2}} \Pi_{N_1, N_2}^{(1)} \cup \bigcup_{(N_1, N_2) \in L_N^*} \Pi_{N_1, N_2}^{(2)}$$

is linearly independent over \mathbb{Q} . The condition $n_1 \leq n_3$ in $\Pi_{N,N}^{(2)}$ can be explained as follows. Let $m = \max\{n_1 \mid (n_1, n_2) \in L_N\}$. The choices of the shuffle product of two Lyndon words of depth 2 and weight N are given by the following.



Let

$$L_{N_1, N_2} = \begin{cases} \{(n_1, \dots, n_4) \mid (n_1, n_2) \in L_{N_1}^*, (n_3, n_4) \in L_{N_2}^*\} & \text{if } N_1 \neq N_2 \\ \{(n_1, \dots, n_4) \mid (n_1, n_2) \in L_{N_1}^*, (n_3, n_4) \in L_{N_2}^*, n_1 \leq n_3\} & \text{if } N_1 = N_2 \end{cases}$$

and

$$\Pi_{N_1, N_2} = \{z_{n_1} \mathbin{\boxplus} z_{n_2} \mathbin{\boxplus} z_{n_3} \mathbin{\boxplus} z_{n_4} \mid (n_1, n_2, n_3, n_4) \in L_{N_1, N_2}\}.$$

Then we easily find that

$$\begin{aligned} \{\alpha_i \mathbin{\boxplus} \beta_j \mid 1 \leq i \leq g, 1 \leq j \leq h\} &= \Pi_{N_1, N_2} \cup \Pi_{N_1, N_2}^{(1)} \cup \Pi_{N_2, N_1}^{(1)} \cup \Pi_{N_1, N_2}^{(2)}, \\ \{\alpha_i \mathbin{\boxplus} \alpha_j \mid 1 \leq i \leq j \leq g\} &= \Pi_{N_1, N_1} \cup \Pi_{N_1, N_1}^{(1)} \cup \Pi_{N_1, N_1}^{(2)}. \end{aligned}$$

For our purpose, it only remains to verify that the set Π_{N_1, N_2} is linearly independent, or equivalently, an overlap arising from the commutativity of the shuffle product $\mathbin{\boxplus}$ doesn't occur. This overlap comes up if we can choose $\sigma \in \mathfrak{S}_4$ such that $\sigma(l) = (n_{\sigma(1)}, \dots, n_{\sigma(4)}) \in L_{N_1, N_2}$ and $\sigma(l) \neq l$ for $l = (n_1, \dots, n_4) \in L_{N_1, N_2}$ (because this σ gives the possible relation $z_{n_1} \mathbin{\boxplus} z_{n_2} \mathbin{\boxplus} z_{n_3} \mathbin{\boxplus} z_{n_4} = z_{n_{\sigma(1)}} \mathbin{\boxplus} z_{n_{\sigma(2)}} \mathbin{\boxplus} z_{n_{\sigma(3)}} \mathbin{\boxplus} z_{n_{\sigma(4)}}$ in Π_{N_1, N_2}). However, we can check that for all $\sigma \in \mathfrak{S}_4$, if $\sigma(l) \in L_{N_1, N_2}$, then we have $\sigma(l) = l$. We show a few cases. Assume $N_1 \leq N_2$. For $l = (n_1, \dots, n_4) \in L_{N_1, N_2}$, one can check that

$$\begin{aligned} \sigma(l) = (n_1, n_2, n_4, n_3) \in L_{N_1, N_2} &\Rightarrow n_4 \leq n_3 \Rightarrow n_3 = n_4, \text{ since } n_3 \leq n_4 \Rightarrow l = \sigma(l), \\ \sigma(l) = (n_1, n_3, n_2, n_4) \in L_{N_1, N_2} &\Rightarrow n_1 + n_3 = n_1 = n_1 + n_2 \Rightarrow n_2 = n_3 \Rightarrow l = \sigma(l), \\ \sigma(l) = (n_3, n_4, n_1, n_2) \in L_{N_1, N_2} &\Rightarrow n_2 \leq n_1 \Rightarrow n_1 = n_2 \Rightarrow n_3 \leq n_1 \text{ and } n_1 \leq n_3 \\ &\Rightarrow n_1 = n_3 \Rightarrow l = \sigma(l), \\ \sigma(l) = (n_4, n_2, n_1, n_3) \in L_{N_1, N_2} &\Rightarrow n_4 + n_2 = n_1 + n_2 \Rightarrow n_1 = n_4 \Rightarrow n_4 = n_1 \leq n_3 \leq n_4 \\ &\Rightarrow n_1 = n_3 = n_4 \Rightarrow l = \sigma(l). \end{aligned}$$

The reminder can be checked in the same way. This completes the proof. \square

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